MATH 7018 - HW 5 (Spring 2013)

Due date: Friday, April 12th

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The first three problems are from Chapter 6 of *Stochastic Processes*, by S. Ross.

Problem 1. Consider a random walk on the set of integers which at each transition either goes up 1 with probability p or down 1 with probability $q = 1 - p$. Argue that $(q/p)^{S_n}$, $n \geq 1$, is a martingale.

Problem 2. Let X_n denote the size of the nth generation of a branching process whose mean number of offspring per individual is m. Note that we may write X_n as follows:

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i \,,$$

using Z_i to represent the number of offspring of the ith individual of the $(n-1)$st generation.

(a) Verify that X_n/m^n, $n \geq 1$, is a martingale.

(b) Let π_0 denote the probability that such a process, starting with a single individual, eventually goes extinct. Show that $\{\pi_0^{X_n}, n \geq 0\}$ is a martingale.

Problem 3. An urn initially contains one white and one black ball. At each stage a ball is drawn and is then replaced in the urn along with another ball of the same color. Let Z_n denote the fraction of balls in the urn that are white after the nth replication.

(a) Show that $\{Z_n, n \geq 1\}$ is a martingale.

(b) Show that the probability that the fraction of white balls in the urn is ever as large as 3/4 is at most 2/3.

Problem 4. (Exercise 7.9.1 from A-S.) Let $G = (V, E)$ be a graph whose vertices are all 7^n vectors of length n over Z_7 (integers mod 7), in which two vertices are adjacent iff they differ in one coordinate. Let $U \subset V$ be a set of 7^{n-1} vertices of G, and let W be the set of all vertices whose distance from U exceeds $(c+2)\sqrt{n}$, where $c > 0$ is a constant. Prove that $|W| \leq 7^n e^{-c^2/2}$.

Problem 5. (Exercise 7.9.3 from A-S.) Prove that there is an absolute constant c such that for every $n > 1$, there is an interval I_n of at most $c\sqrt{n}/\log n$ consecutive integers such that the chromatic number of the random graph satisfies:

$$\Pr[\ \chi(G(n, 1/2) \in I_n\ ] \geq 0.99\,.$$