MATH 7018 - Homework 1 (Spring 2014)

Due date: Thursday, January 23rd
Instructor: Prasad Tetali

Please submit organized and well written solutions

Some of the exercises are from the Alon-Spencer textbook, as indicated in parantheses.

Problem 1. Recall that $m(n)$ is the smallest size family of n-sets which is *not* 2-colorable. Prove that $m(3) > 6$, by showing that every collection of 6 or fewer 3-sets is 2-colorable.

Hint. Use the probabilistic method: first show that the ground set might as well be of size at most 9. Then consider a *uniform random equi-coloring* – over all possible 2-colorings with 5 Reds and 4 Blues, choose one at random, uniformly.

Problem 2. (Exercise 1.6.1) Prove that if there is a real p, $0 \le p \le 1$, such that

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1\,,$$

then the Ramsey number $R(k,t)$ satisfies $R(k,t) > n$. Using this show that

$$R(4,t) \ge \Omega\Big(t^{3/2}/(\ln t)^{3/2}\Big)\,.$$

Problem 3. (Exercise 1.6.10) Prove that there is an absolute constant $c > 0$ with the following property; Let A be an $n \times n$ matrix with pairwise distinct entries. Then there is a permutation of the rows of A so that no column in the permuted matrix contains an increasing subsequence of length at least $c\sqrt{n}$.

Problem 4. (Exercise 1.6.4) Let $G = (V, E)$ be a graph with n vertices and minimum degree $\delta > 10$. Prove that there is a partition of V into two disjoint subsets A and B so that $|A| \le O(n \ln \delta/\delta)$ and each vertex of B has at least one neighbor in A and at least one neighbor in B.

Problem 5. (Exercise 2.7.7) Let $\mathcal{F}$ be a family of subsets of $N = \{1, 2, \dots, n\}$, and suppose that there are no $A, B \in \mathcal{F}$ satisfying $A \subset B$. Let $\sigma \in S_n$ be a random permutation of the elements of N and consider the random variable

$$X = \mid \{i : \{\sigma(1), \sigma(2), \dots, \sigma(i)\} \in \mathcal{F}\} \mid .$$

By considering the expectation of X prove that $\mid \mathcal{F} \mid \le \binom{n}{\lfloor n/2 \rfloor}$.

Problem 6. (a) Show that there exists an $n \times n$ matrix (for all n) with entries from $\{+1, -1\}$ whose determinant is at least $\sqrt{n!}$.

(b) What is the determinant of a Hadamard matrix of size $n \times n$? [Recall that a *Hadamard matrix* is a square matrix with entries $+1, -1$, and with the property that the rows vectors are mutually orthogonal (and hence also the column vectors).]

Problem 7. Prove that every set A of n nonzero integers contains two *disjoint* subsets $B_1, B_2 \subseteq A$, so that $|B_1| + |B_2| > 2n/3$ and each set B_i is sum-free (that is, there are no $b_1, b_2, b_3 \in B_i$ so that $b_1 + b_2 = b_3$.)

Optional Subproblems: (No need to submit)

Problem O1. (Exercise 1.6.2) Suppose $n > 4$ and let H be an n-uniform hypergraph with at most $4^{n-1}/3^n$ edges. Prove that there is a coloring of the vertices of H by 4 colors so that in every edge all 4 colors are represented.

Problem O2. (Exercise 1.6.8) Let F be a finite collection of binary strings of finite lengths and assume no member of F is a prefix of another one. Let n_i denote the number of strings of length i in F. Prove that

$$\sum_i \frac{n_i}{2^i} \leq 1 \, .$$

Problem O3. We say a subset A of a set S of distinct positive integers is *sum-avoiding* if the sum of any distinct pair of integers from A is *not contained in* S. That is, $\forall a, b \in A$, we need $a + b \notin S$. Show that any set of n distinct positive integers contains a sum-avoiding subset of size at least $\log n$. *Hint: No need for any probability; but use the Turán-type bound on the independence number of a graph we saw in class :* $\alpha(G) \geq \sum_v \frac{1}{d(v)+1}$.