MATH 7018 - HW 2 (Spring 2014)

Due date: Thursday, Feb. 13th
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Problem 1. Let X be the number of cycles (not necessarily induced) in $G(n,p)$. Show that if $p = o(n^{-1})$ then $\Pr[X > 0] = o(1)$. Let $c \in (0,1)$ be fixed. Show that with $p = c/n$, $E[X] = O(1)$, where the implicit constant in $O(\cdot)$ only depends on $c > 0$.

Problem 2. (Exercise 4.8.2) Show that there is a positive constant c such that the following holds. For any n reals $a_1, a_2, \ldots, a_n$ satisfying $\sum_i a_i^2 = 1$, if $(\epsilon_1, \ldots, \epsilon_n)$ is a $\{-1,1\}$-random vector obtained by choosing each ϵ_i randomly and independently with uniform distribution to be either -1 or +1, then

$$\Pr\Big[|\sum_{i=1}^{n} \epsilon_i a_i| \leq 1\Big] \geq c\,.$$

(*Hint.* Consider two cases – either there is an a_i with absolute value at least 1/2 or not – and argue separately.)

Problem 3. (Exercise 4.8.4) Let X be a random variable with expectration $E[X] = 0$ and variance σ^2. Prove that for all $\lambda > 0$,

$$\Pr[X \geq \lambda] \leq \frac{\sigma^2}{\sigma^2 + \lambda^2}\,.$$

Problem 4. (Exercise 4.8.5) For $1 \leq i \leq n$, let $v_i = (x_i, y_i)$ be n two-dimensional vectors, where each x_i and y_i is an integer whose absolute value does not exceed $2^{n/2}/(100\sqrt{n})$. Show that there are two disjoint sets $I, J \subset \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in I} v_i = \sum_{j \in J} v_j\,.$$

Reading Exercise. Read the Probabilistic Lens on "Weierstrass Approximation Theorem" on Pages 117-118 of [A-S]. The proof (due to Bernstein) is elementary and uses Chebyshev's inequality on the binomial random variable.