MATH 7018 - HW 3 (Spring 2014)

Due date: Thursday, March 13th
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Problem 1. Let $X_1, X_2, \ldots$ be independent random variables, each *exponentially distributed* with rate 1: for $n \geq 1$,

$$\Pr(X_n > x) = e^{-x}\,, \quad x \geq 0\,.$$

Show that

$$\Pr(X_n > \log n + \alpha \log\log n \text{ i.o.}) = 0, \text{ if } \alpha > 1, \text{ and } = 1, \text{ if } \alpha \leq 1\,.$$

Problem 2. Let $X_1, X_2, \ldots$ be i.i.d. random variables. Show that

$$\Pr[|X_n| \geq n \text{ i.o.}] = 0,$$

if and only if $\mathrm{E}[|X_1|] < \infty$.

Problem 3. Let $\{S_n | n \geq 0\}$ denote the position of the simple random walk, which moves to the right with probability p and to the left with probability $q = 1 - p$ at each step on the integer line. Suppose that $S_0 = 0$. Show that

$$\Pr[S_n = 0 \text{ i.o.}] = 0, \quad \text{if } p \neq 1/2\,.$$

Problem 4. Define $f(k)$ to be the least n so that if the subsets S of $\{1, 2, \ldots, n\}$ of size k are two-colored then there exists a set $T \subset \{1, 2, \ldots, n\}$ of size $k+1$, all of whose k–element subsets are of the same color. (The existence of such an n follows from Ramsey's theorem, but here we are concerned with *lower bounds*.) Use both the basic probabilistic method and the Lovász local lemma to find lower bounds on $f(k)$.

Problem 5. (a) Prove the following version of the local lemma. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. Consider as usual a set of n events, $A_1, A_2, \ldots, A_n$; each event A_i also has an associated $D_i \subset [n]$, such that A_i is mutually independent of events A_j, for $j \in [n] \setminus D_i$. If for each $i \in [n]$,

(i) $\Pr(A_i) \leq 1/8$, and

(ii) $\sum_{j \in D_i} \Pr(A_j) \leq 1/4$,

then with positive probability, none of the events happen.

(b) For $\beta \geq 1$, a proper coloring of the vertices of a graph is called a β-capped proper coloring, if for each vertex v and color c, the number of times c appears in the neighborhood of v is at most β. If G has maximum degree $\Delta \geq \beta^\beta$, then G has a β-capped proper vertex coloring using at most $16\Delta^{1+1/\beta}$ colors.

Problem 6. Let $A_1, A_2, \ldots$ be a sequence of events. Let $B_n = \cup_{m=n}^{\infty} A_m$ and $C_n = \cap_{m=n}^{\infty} A_m$. Clearly, $C_n \subset A_n \subset B_n$, with B_n: decreasing and C_n: increasing sequences of events with the limits:

$$B = \lim B_n = \cap B_n = \cap_n \cup_{m \geq n} \ A_m =: \limsup_{n \to \infty} \ A_n \, ,$$
$$C = \lim C_n = \cup C_n = \cup_n \cap_{m \geq n} \ A_m =: \liminf_{n \to \infty} \ A_n \, .$$

Show that

(a) $B = \{\omega \in \Omega : \omega \in A_n$ for *infinitely* many $n\}$.

(b) $C = \{\omega \in \Omega : \omega \in A_n$ for all but finitely many $n\}$.

If $B = C$, we say $A_n \to A$ where $A = B = C$. In such a case, show that

(c) A is an event, i.e., that $A \in \mathcal{F}$, and that

(d) $\Pr(A_n) \to \Pr(A)$, as $n \to \infty$.

Optional Problems.

Problem O1. Let $\mathcal{F}$ be a σ-field of subsets of Ω. Suppose $B \in \mathcal{F}$. Show that $\mathcal{G} = \{A \cap B : A \in \mathcal{F}\}$ is a σ-field of subsets of B. (As you can imagine, this property is convenient in defining a *conditional* probability measure over such a $\mathcal{G}$, conditioned on an event B of positive probability.)

Problem O2. Given a collection of sets: $\mathcal{A} = \{A_1, \ldots, A_n\}$, with each $A_i \in \Omega$, show that it can be extended to a σ-field as follows: Let

$$\sigma(\mathcal{A}) := \cap_{\hat{\sigma}} \ \hat{\sigma}(A_1, A_2, \ldots, A_n) \, ,$$

where the intersection is over all $\hat{\sigma}$: σ-fields containing all A_i. Argue that $\sigma(\mathcal{A})$ is the smallest σ-field containing $A_1, A_2, \ldots, A_n$.

Problem O3. Let $\mathcal{F}$ be a σ-field over subsets of Ω. Suppose $P : \mathcal{F} \to [0, 1]$ satisfies: $P(\phi) = 0$, $P(\Omega) = 1$, and

$$P(A \cup B) = P(A) + P(B) \, , \text{ for all } A, B \in \mathcal{F}, \text{ with } A \cap B = \phi \, .$$

Show that if P is *continuous*, in the sense discussed in class, then P is countably additive.