# MATH 7018 - HW 4 (Spring 2014)

**Due date: Thursday, April 10th**
**Instructor: Prasad Tetali**

**Problem 1.** (Strong Law under stronger moment hypothesis). Suppose that $X_1, X_2, \ldots,$ are independent r.v.s with mean 0, and so that

$$\mathrm{E}(X_n^4) \leq C, \quad \forall n \geq 1\,,$$

for some constant $C > 0$. Show that, for $S_n := \sum_{i=1}^n X_i$, we have

$$\Pr(S_n/n \to 0) = 1\,.$$

*Hint.* First show that $\mathrm{E}(S_n^4) \leq 3Cn^2$.

**Problem 2.** A bag contains red and green balls. A ball is drawn from the bag, its color noted, then it is returned to the bag together with a new ball of the same color. Initially the bag contained one ball of each color, say. If $G_n$ denotes the number of green balls in the bag after $n$ additions, show that $S_n = G_n/(n+2)$ is a martingale. Deduce that the ratio of the green to red balls converges almost surely to some limit as $n \to \infty$.

**Problem 3.**
(a) For $X, Y$: r.v.s, verify that

$$\mathrm{Var(Y) = E[Var(Y|X)] + Var(E[Y|X])}\,.$$

(b) Let $N$ be a nonnegative integer r.v. Let $S$ be the sum of a *random* number of r.v.s, defined as $S := \sum_{i=1}^N X_i$, with $X_i$ being i.i.d. Then show that

$$\mathrm{E[S] = E[N]E[X_1]}, \text{ and that } \mathrm{Var(S) = Var(N)(E[X_1])^2 + Var(X_1)E[N]}\,.$$

**Problem 4**. (Exercise 7.9.1 from A-S.) Let $G = (V, E)$ be a graph whose vertices are all $7^n$ vectors of length $n$ over $Z_7$ (integers mod 7), in which two vertices are adjacent iff they differ in one coordinate. Let $U \subset V$ be a set of $7^{n-1}$ vertices of $G$, and let $W$ be the set of all vertices whose distance from $U$ exceeds $(c+2)\sqrt{n}$, where $c > 0$ is a constant. Prove that $|W| \leq 7^n e^{-c^2/2}$.

**Problem 5.** (Exercise 7.9.2 from A-S.) Let $G = (V, E)$ be a graph with chromatic number $\chi(G) = 1000$. Let $U \subset V$ be a random subset of $V$ chosen uniformly among all $2^{|V|}$ subsets of $V$. Let $H = G[U]$ be the induced subgraph of $G$ on $U$. Prove that

$$\Pr[\chi(H) \leq 400] < 1/100\,.$$