MATH 7018 - TEST 1 (Spring 2014)

Due date: Thursday, Feb. 20th
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Please try to work on your own – it's the only difference from the hws!

Problem 1. For a graph G and a subset T (for TOP) of the vertices, call an edge *crossing* if it has exactly one vertex in T and let $N(G;T)$ denote the number of such edges.

Prove for an appropriate (i.e., as small as possible) $\alpha = \alpha(n)$ that there exists a graph G of $2n$ vertices so that for every set T of n vertices,

$$\mid N(G;T) - \frac{n^2}{2} \mid < \alpha.$$

Problem 2. Prove that every three-uniform hypergraph with n vertices and $m \geq n/3$ edges contains an independent set of size at least $\frac{2n^{3/2}}{3\sqrt{3}\sqrt{m}}$.

Problem 3. Let $G = (V, E)$ be a bipartite graph on n vertices with a list $S(v)$ of more than $\log_2 n$ colors associated with each vertex $v \in V$. Prove that there is a proper coloring of G assigning to each vertex v a color from its list $S(v)$.

Problem 4. (Weaker bound for Property B.) Show that every n-uniform hypergraph can be properly 2-colored as long as the number of edges is $O(2^n n^{1/3} (\ln n)^{-1/2})$.

Hint: First color each element randomly red/blue (with equal probability and independently). Then flip the color, with probability p, of each element involved in a monochromatic set from the first coloring. To bound the probability that edge e was red in the first coloring and f was blue in the recoloring, bound the probability of the event A_{efR} where $R \subset f \setminus e$ is the subset of f (besides $e \cap f$ which turned red to blue).

Problem 5. Let $[n]_p$ denote a random subset of integers chosen from the set $[n] := \{1, 2, \ldots, n\}$ as follows: $p = p(n)$ is the probability of including each integer from $[n]$, independently of the others. Determine the threshold function (in the sense of Section 4.4 of [Alon-Spencer]) for the existence of a 3-term arithmetic progression (3TAP) in $[n]_p$:

That is, identify a function $r(n)$ such that if $p(n) \gg r(n)$, then the probability that $[n]_p$ contains a 3TAP goes to 1 (as $n \to \infty$), while if $p(n) \ll r(n)$, the probability goes to 0, as $n \to \infty$.