Probabilistic Combinatorics

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Tentative Plans

1. Basics for Probability theory

Coin Tossing, Expectation, Linearity of Expectation, Probability vs. Expectation, Bool's Inequality

2. First moment (or expectation) method and Markov Inequality

Applications to problems regarding

- Property B
- Arithmetic Progressions
- Covering hypercubes
- Ramsey numbers
- Independence numbers of graphs
- Occupancy Problems

3. Methods of conditional events

- Property B & Recoloring (Beck's theorem)
- Independent numbers of sparse graphs (AKS* theorem)
- Covering hypercubes (K & Roche's theorem)

*Ajtai, Komlós and Szemerédi

4. Second Moment Method (or Chebyschev's Inequality)

Applications to problems regarding

- Arithmetic progression
- Random graphs
- Perfect matchings in random uniform hypergraphs
- Covering hypercube
- Occupancy Problems and Poisson approximation

4. Law of large numbers

- Chernoff Bounds
- Martingale Inequalities
- Talagrand Inequality,

Applications to problems regarding:

- Ramsey numbers
- Chromatic number of $G(n, 1/2)$,
- Incremental random method
- Cut-off line Algorithm:

  Matching of uniform random numbers $\in [0, 1]$

6. Lovasz Local Lemma

7. Incremental random methods

8. Branching Processes

9. Poisson Cloning Model

10. Random regular graph and contiguity

1 Basics for Probability Theory

1.1 Probabilities, Events and Random variables

- One coin tossing

$$X = \begin{cases} 1 & \text{if "HEAD"} \\ 0 & \text{if "TAIL"} \end{cases}$$

Thus

$$\Pr[X = 0] = \Pr[X = 1] = 1/2.$$

- Two coin tossing: X_1, X_2

$$\Pr[X_1 = 0, X_2 = 0] = \Pr[X_1 = 1, X_2 = 0]$$
$$= \Pr[X_1 = 0, X_2 = 1] = \Pr[X_1 = 1, X_2 = 1] = 1/4$$

- n coin tossing: $X_1, X_2, ..., X_n$

For any $(x_i) \in \{0,1\}^n$,

$$\Pr[X_1 = x_1, X_2 = x_2, ..., X_n = x_n] = 1/2^n.$$

We say that

$X_1, ..., X_n$: independent and identically distributed
or simply i.i.d

- Event

· Event A:

$$\text{10th coin is ``HEAD''}$$

or

$$A = \{X_{10} = 1\}.$$

· Event B:

$$\#\text{ of heads} = k$$

or

$$B = \{X_1 + \cdots + X_n = k\}.$$

- Probability of events

$$\Pr[X_{10} = 1] = 1/2 \quad \text{or} \quad \Pr[A] = 1/2$$

$$\Pr[X_1 + \cdots + X_n = k] = \binom{n}{k} 2^{-n} \quad \text{or} \quad \Pr[B] = \binom{n}{k} 2^{-n},$$

where $\binom{n}{k}$=the # ways of to take k objects out of n objects

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \text{“}n \text{ choose } k\text{”}$$

- n biased coin tossing:

$$X_1, ..., X_n : \text{i.i.d.}$$

$$\Pr[X_i = 0] = 1 - p \quad \text{and} \quad \Pr[X_i = 1] = p$$

- Events & Probabilities

$$\Pr[X_{10} = 1] = p$$

$$\Pr[X_1 + \cdots + X_n = k] = \binom{n}{k} p^k (1-p)^{n-k}.$$

• Properties

For any two events A, B,

(a) If $A \subseteq B$, then

$$\Pr[A] \leq \Pr[B].$$

E.g.

$$A = \{X_5 = 1, X_{11} = 0\}, \quad B = \{X_{11} = 0\}$$

Then $A \subseteq B$ and

$$p(1-p) = \Pr[A] \leq \Pr[B] = 1 - p.$$

(b) For any two events, A and B

$$\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B].$$

In particular,

$$\Pr[A \cup B] \leq \Pr[A] + \Pr[B].$$

Generally,

(**Boole's Inequality**) For events $A_1, ..., A_m$,

$$\Pr[\cup_{i=1}^{m} A_i] \leq \sum_{i=1}^{m} \Pr[A_i].$$

• Random variables (RV)

E.g.

· $S = X_1 + \cdots + X_n$ where $X_1, ..., X_n$: i.i.d. and

$$\Pr[X_1 = 0] = 1 - p \;\; \text{and} \;\; \Pr[X_1 = 1] = p.$$

· k (indistinguishable) balls and n bins

Each ball is to be distributed uniformly at random

so that

$$\Pr[\; i^{\text{th}} \text{ ball is in } j^{\text{th}} \text{ bin }] = 1/n$$

Define RV

$$X = \# \text{ of balls in first 10 bins}$$

- Expectation of nonnegative integral valued RV X:

$$E[X] := \sum_{k=0}^{\infty} k \Pr[X = k].$$

E.g. $S = X_1 + \cdots + X_n$

$$\begin{aligned} E[S] &= \sum_{k=1}^{\infty} k \Pr[S = k] \\ &= \sum_{k=1}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} = pn \end{aligned}$$

Easy: $\Pr[X > 0] \leq E[X]$,

$$\Pr[X > 0] = \sum_{k=1}^{\infty} \Pr[S = k] \leq \sum_{k=1}^{\infty} k \Pr[S = k] = E[X].$$

• Linearity of Expectation

For any RV's X and Y,

$$E[X+Y]=E[X]+E[Y].$$

More generally, for RV's $X_1, ..., X_n$,

$$E[X_1+\cdots+X_n]=E[X_1]+\cdots+E[X_n].$$

E.g.

$$\begin{aligned} E[S]=E[X_1+\cdots+X_n] &= E[X_1]+\cdots E[X_n] \\ &= p+\cdots+p=pn. \end{aligned}$$

2 First Moment (Expectation) Method

For nonnegative integral valued RV X

$$\Pr[X > 0] = E[1(X > 0)] \leq E[X1(X > 0)] = E[X],$$

in particualr, if $E[X] < 1$, then

$$\Pr[X = 0] \geq 1 - E[X] > 0.$$

On the other hand,

$$E[X] \geq k \Longrightarrow \Pr[X \geq k] > 0.$$

so that

$$\text{there is an instance which makes } X \geq k$$

2.1 Property B

A hypergraph $H = (V, E)$ has Property B, or 2-colorable,
if $\exists$ a 2-coloring of V s.t.
no edge in H is monochromatic.

Theorem If a k-uniform hypergraph H has less than 2^{k-1} edges, then H has property B.

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Proof. Color each vertex, randomly and independently, either B (blue) or R (red) with equal probability. For $e \in H$,

$$\Pr[e \text{ is monochromatic}] = 2^{-k+1} \quad \text{yields}$$

$$E[\# \text{ of monochromatic edges}] = |H| 2^{-k+1} < 1,$$

and hence

$$\Pr[\exists \text{ no monochromatic edge}] > 0.$$

Thus

$$\exists \text{ 2-coloring with no monochromatic edge.}$$

$\square$

2.2 Arithmetic progressions and van der Waerden number $W(k)$

· Arithmetic Progression (AP) with k terms in $\{1, ..., n\}$

$$a, a + d, a + 2d, ..., a + (k - 1)d \in \{1, ..., n\}$$

Let $W(k)$ be the least n so that, if $\{1, ..., n\}$ is two-colored,

$$\exists \text{ a monochromatic AP with } k \text{ terms}$$

$$W(3) = 9, W(4) = 35, W(5) = 178, ...$$

van der Waerden ('27)

$$W(k) \text{ is FINITE for any } k.$$

Theorem

$$W(k) \geq 2^{k/2}.$$

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Proof. Two-color $\{1, ..., n\}$ randomly, say Red & Blue so that

$$\Pr[i \text{ is colored Red}] = \Pr[i \text{ is colored Blue}] = 1/2$$

independently of all $j \neq i$. For each k-term AP S in $\{1, ..., n\}$,

$$\Pr[S \text{ is monochromatic }] = 2^{1-k}.$$

Since there are at most $n^2/2$ such S (WHY?), if

$$n < 2^{k/2}$$

then

$$E[\# \text{ of monochromatic } S] \leq (n^2/2)2^{1-k} < 1,$$

and

$$\Pr[\exists \text{ no monochromatic } S] > 0.$$

$\square$

2.3 Covering n-cube

n-cube Q_n :

$$\{-1,1\}^n = \{(x_{_i}) : x_{_i} = 1 \text{ or} -1, i = 1, ..., n\}$$

Let $X_1, ..., X_m$ be (mutually) independent uniform random vectors in Q_n, in particular,

$$\Pr[X_j = u] = 2^{-n} \;\text{ for any }\; u \in Q_n.$$

Theorem If $m = (1+\varepsilon)n$ for $\varepsilon > 0$, then

$$\Pr[\exists\, w \in Q_n \text{ with } w \cdot X_j > 0 \;\; \forall j = 1, ..., m] \le 2^{-\varepsilon n} \longrightarrow 0,$$

as $n \to 0$.

Theorem If $m = (1+\varepsilon)n$ for $\varepsilon > 0$, then

$$\Pr[\exists\, w \in Q_n \text{ with } w \cdot X_j > 0 \ \ \forall j = 1, ..., m] \leq 2^{-\varepsilon n} \longrightarrow 0,$$

as $n \to 0$.

Proof. For $w \in Q_n$, let Y_w be the indicator RV for the event A_w that $w \cdot X_j > 0$ for all $j = 1, ..., m$. Then

$$E[\#\text{ of } w \text{ with } w \cdot X_j > 0 \ \forall j = 1, ..., m] = E[\sum_{w \in Q_n} Y_w] = \sum_{w \in Q_n} \Pr[A_w].$$

As X_j's are mutually independent,

$$\Pr[A_w] = \Pr[w \cdot X_j > 0 \text{ for all } j = 1, ..., m] = \prod_{j=1}^{m} \Pr[w \cdot X_j > 0] \leq (1/2)^m.$$

Therefore, the probability is bounded by $2^{n-m} = 2^{-\varepsilon n}$.

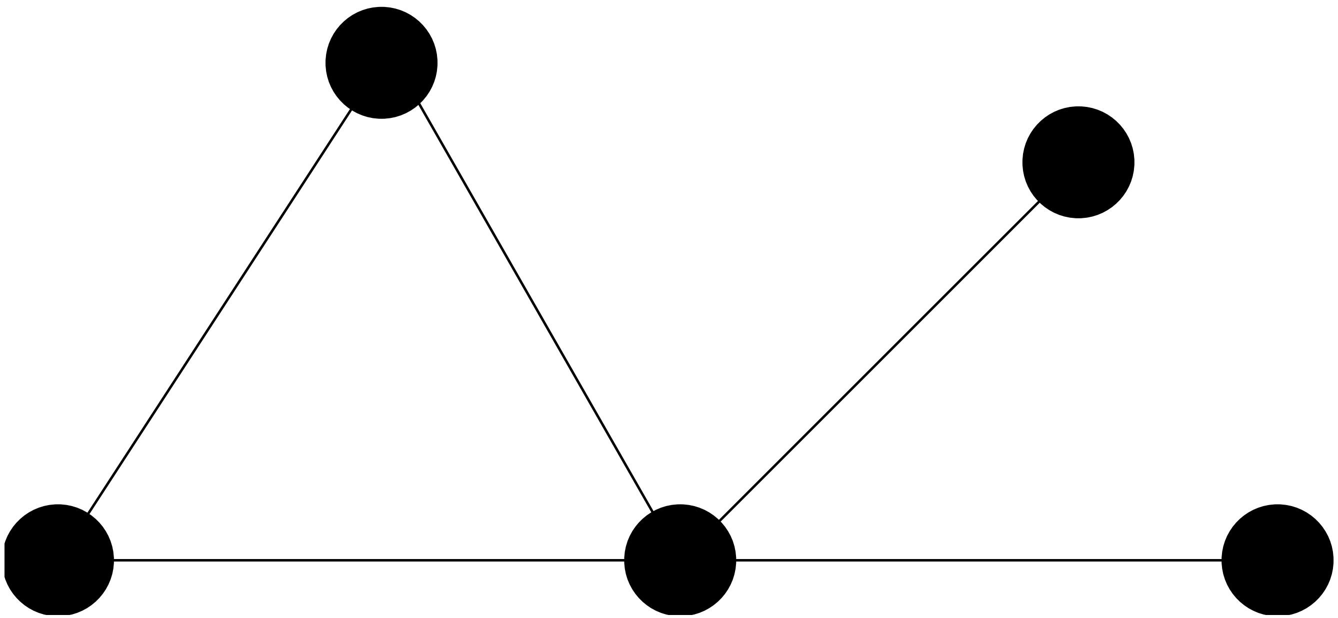
$\square$

2.4 Ramsey Number $R(s, t)$

Recall, for a graph G,

$\omega(G)$: clique number of G (size of a largest clique)

$\alpha(G)$: independence number of G (size of a largest independent set)



$$R(s,t) \quad := \quad \min\{n : \text{for every } G \text{ on } n \text{ vertices,}$$
$$\omega(G) \geq s \text{ or } \alpha(G) \geq t\}$$

EASY

$$R(s,t) = R(t,s), \qquad R(2,t) = t$$

Greenwood & Gleason ('55):

$$R(3,3) = 6, \quad R(3,4) = 9,$$

$$R(3,5) = 14, \; R(4,4) = 18$$

MORE:

$$R(3,6) = 18, \; R(3,7) = 23, \; R(3,8) = 28,$$
$$R(3,9) = 36, \; R(4,5) = 25,$$

$$43 \leq R(5,5) \leq 49$$

There are

$$2^{\binom{n}{2}}$$

graphs on n vertices, where

$$\binom{n}{2} = n(n-1)/2 \ .$$

For example, if $n = 28$

$$\begin{aligned} 2^{\binom{28}{2}} &\approx 6156563648\cdots\cdots \\ &\approx 0.6156563648\cdots\cdots \times 10^{114} \end{aligned}$$

Ramsey('30): $R(s,t)$ is FINITE.

Skolem('33), Erdős and Szekeres('35):

$$R(s,t) \le R(s-1,t) + R(s,t-1)$$

and

$$R(s,t) \le \binom{s+t-2}{s-1}$$

Theorem. If
$$\binom{n}{t} \cdot 2^{1-\binom{t}{2}} < 1$$
then
$$R(t,t) > n.$$

Proof. Random graph $G = G(n, 1/2)$:

Each of $\binom{n}{2}$ edges in K_n is in G
with probability $1/2$, independently of all other edges

For each subset T of size t, let A_T be the event that T is a clique in G. Then
$$\Pr[\omega(G) \geq t] \leq \Pr[\bigcup_T A_T] \leq \sum_T \Pr[A_T].$$

Since T has $\binom{t}{2}$ edges in it,
$$\Pr[A_T] = 2^{-\binom{t}{2}}$$

and

$$\Pr[\omega(G) \geq t] \leq \binom{n}{t} 2^{-\binom{t}{2}}.$$

Similarly,

$$\Pr[\alpha(G) \geq t] \leq \binom{n}{t} 2^{-\binom{t}{2}}.$$

Thus

$$\begin{aligned}
\Pr[\omega(G) \geq t \text{ or } \alpha(G) \geq t] &\leq \Pr[\omega(G) \geq t] + \Pr[\alpha(G) \geq t] \\
&\leq \binom{n}{t} 2^{1-\binom{t}{2}} < 1,
\end{aligned}$$

which implies that

$$\Pr[\omega(G) < t \ \text{ and } \ \alpha(G) < t] > 0,$$

in particular, there is at least one such graph.

$\square$

Using Stirling formula,

$$n! = \sqrt{2\pi n}\, e^{\varepsilon_n} \left(\frac{n}{e}\right)^n,$$

where $1/(12n+1) < \varepsilon_n < 1/(12n)$,

$$R(t,t) \geq n = (1+o(1))(t/e)2^{(t-1)/2} = \frac{(1+o(1))t}{e\sqrt{2}} 2^{t/2}.$$

Hence

$$\frac{(1+o(1))t}{e\sqrt{2}} 2^{t/2} \leq R(t,t) \leq \binom{2t-2}{t-1} \sim c4^t/\sqrt{t}.$$

BIG open problem:

$$\lim_{t\to\infty} R(t,t)^{1/t} = ???$$

Even existence is not known.

$$\begin{aligned}\sqrt{2} &\le \liminf R(s,s)^{1/s} \\ &\le \limsup R(s,s)^{1/s} \le 4\end{aligned}$$

Theorem If

$$\binom{n}{s}p^{\binom{s}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1$$

for some $0 \le p \le 1$, then

$$R(s,t) > n.$$

Proof. Exercise (take each edge with probability p.)

$\square$

For fixed s,

$$R(s,t) \ge c_s \left(\frac{t}{\log t}\right)^{(s-1)/2}.$$

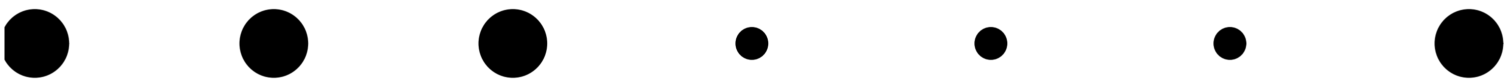
2.5 Independence numbers of graphs

Theorem (Turán) For a graph $G = (V, E)$ with $|V| = n$ and the average degree $t(G) = \frac{1}{n}\sum_{v \in V} d(v)$

$$\alpha(G) \geq \frac{n}{t(G)+1}.$$

(A probabilistic) Proof.

Randomly order all vertices of G,



Take the first vertex v_1 and delete all verts in its nbd $N(v)$.

Take the next undeleted vertex and do the same.

Let I be the independent set obtained. Enough to show that

$$E[\|I\|] \geq \frac{n}{t+1} \ ,$$

Note that

$$E[\|I\|] = E[\sum_v 1(v \in I)] = \sum_v \Pr[v \in I] \ .$$

where

$$1(v \in I) = \begin{cases} 1 & \text{if } v \in I \\ 0 & \text{if } v \notin I. \end{cases}$$

If a vertex v precedes all verts in $N(v)$ then

$$v \in I.$$

The corresponding probability is

$$\frac{1}{d(v)+1} \ .$$

We have

$$\Pr[v \in I] \geq \frac{1}{d(v)+1} \ .$$

Thus

$$E[|I|] \geq \sum_{v} \frac{1}{d(v)+1} \geq \frac{n}{(1/n)\sum_v d(v)+1} = \frac{n}{t+1} \ .$$

by Jensen's Ineq.. $\square$

2.6 Markov Inequality

Theorem 1 *(Markov Inequality) For a random variable* $X \geq 0$,

$$\Pr[X \geq \lambda] \leq \frac{E[X]}{\lambda}.$$

Proof.

$$\Pr[X \geq \lambda] \leq E[(X/\lambda)1(X \geq \lambda)] \leq E[X/\lambda] = E[X]/\lambda,$$

where

$$1(X \geq \lambda) = \begin{cases} 1 & \text{if } X \geq \lambda \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$E[1(X \geq \lambda)] = \Pr[X \geq \lambda].$$

2.7 Occupancy Problems

Occupancy Problems: Insert each of m balls to n distinct bins uniformly at random. Two question:

- What is the maximum number of balls in any bin?
- What is the expected number of bins containing k balls in them.

Consider the case $m = n$. For all $i = 1, 2, \cdots, n$,

$$X_i = \# \text{ of balls in the } i\text{-th bin.}$$

Then

$$E[X_i] = ??.$$

Note that

$$Pr[X_i = 1] = \binom{n}{1}(1/n)(1 - 1/n)^{n-1},$$

in general,

$$Pr[X_i = j] = \binom{n}{j}(1/n)^j(1 - 1/n)^{n-j}.$$

Thus

$$E[X_i] = \sum_{j=1}^{n} \binom{n}{j}(1/n)^j(1 - 1/n)^{n-j} = 1.$$

Or, since $\sum_{i=1}^{n} X_i = n$, in particular, $\sum_{i=1}^{n} E[X_i] = n$, and all $E[X_i]$'s are the same, $E[X_i] = 1$.

Recall $m = n$ and X_i is the number of balls in the i-th bin, $i = 1, 2, \cdots, n$.

For which value k does no bin receive
more than k balls with high probability?

Since

$$\Pr[X_i = j] = \binom{n}{j}(1/n)^j(1-1/n)^{n-j},$$

and Stirling formula gives

$$j! = (1 + O(\tfrac{1}{j}))\sqrt{2\pi j}\left(\frac{j}{e}\right)^j,$$

we have

$$\Pr[X_i \geq \ell] \leq (1 + O(\tfrac{1}{\ell}))\sum_{j \geq \ell}(e/j)^j \leq c(e/\ell)^\ell.$$

Thus, $(e/\ell)^\ell \leq n^{-2}$ implies that

$$\Pr[\exists\ X_i \geq \ell] \leq \sum_i \Pr[X_i \geq \ell] \leq n \cdot cn^{-2} \ll 1.$$

Theorem With probability at least $1 - O(1/n)$, no bin has more than $\lceil (e \ln n)/\ln\ln n \rceil$ balls in it.

Exercise For $m = n \log n$, show that with probability $1 - o(1)$ every bin contains $O(\log n)$ balls.

3 Methods of conditional events

3.1 Property B & Recoloring

Recall that

a hypergraph $H = (V, E)$ has Property B, or 2-colorable,
if $\exists$ a 2-coloring of V
s.t. no edge in H is monochromatic.

Have proved: Any k-uniform hypergraph with $|H| < 2^{k-1}$ have property B.

Theorem (Beck '78) If a k-uniform hypergraph H has at most

$$ck^{1/3}(\ln k)^{-1/2}2^{k-1}$$

edges, then H has property B for any constant $c < \sqrt{3}$.

Remark:

Radhakrishnan & Srinivasan ('99)

If

$$|H| \leq 0.7\Big(\frac{k}{\ln k}\Big)^{1/2} 2^k$$

then then H has property B.

On the other hand,

$$\exists \text{ non-2-colorable } k\text{-uniform}$$
$$\text{hypergraph with } ck^2 2^k \text{ edges}$$

Proof. Ex. (Take random $m = ck^2 2^k$ k-subsets of $\{1, ..., k^2\}$ and use the first moment method.)

Let

$$f(k) = \min\{m : \exists \text{ non-2-colorable } k\text{-uniform hypergraph with } m \text{ edges }\}.$$

Then

$$0.7\left(\frac{k}{\ln k}\right)^{1/2} 2^k \le f(k) \le ck^2 2^k.$$

Proof. Let $p = \ln k/(3k)$ and $t = ck^{1/3}(\ln k)^{-1/2}$, where c is a constant $< \sqrt{3}$. Then

$$2te^{-pk} + t^2 pe^{pk} < 1.$$

We need to show that if

$$|H| \leq t2^{k-1}$$

then H has property B.

Color each vertex, randomly and independently, either B (blue) or R (red) with equal probability (as before).

Call this "first coloring"

Let W be the set all vertices which belong to at lease one monochromatic edge.

Independently change the color of each $v \in W$ with probability p

Call this "recoloring"

An edge e after recoloring is monochromatic by two reasons:

1. Event A_e : e is monochromatic in both the first coloring and the recoloring

$$\begin{aligned}
\Pr[A_e] &= \Pr[e \text{ monoch. in first col. }] \\
&\quad \cdot \Big(\Pr[\text{ no color in } e \text{ is changed (in the recol.)}] \\
&\qquad + \Pr[\text{ all colors in } e \text{ is changed (in the recol.)}] \Big) \\
&= 2^{1-k}((1-p)^k + p^k) \\
&\leq 2^{1-k}(2(1-p)^k) \leq 2^{2-k} e^{-pk}
\end{aligned}$$

and

$$\Pr[\exists \text{ such } e] \leq |H| 2^{2-k} e^{-pk} \leq 2te^{-pk}.$$

2. Event B_e : $\exists \emptyset \neq U_0 \subseteq e$ s.t.

$$\text{in the first col, } e \setminus U_0 \text{ was red and } U_0 \text{ was blue,}$$

and

$$\text{in the recol. all col. in } e \setminus U_0 \text{ remains the same and}$$
$$\text{all col. in } U_0 \text{ are changed,}$$

or the same with red/blue reversed.

Notice that this case also requires at least one edge, say f, with

$$f \cap e \neq \emptyset, \; f \cap e \subseteq U_0, \; \text{and } f \text{ is blue monoch.}$$

Let $U = U_0 \setminus f$. Then the case implies that $\exists f, U$(possibly $\emptyset$) with $e \cap f \neq \emptyset$ and $U \subseteq e \setminus f$ s.t. the event B_{efU} that

$$\text{in the first col., } e \setminus (U \cup f) \text{ was red, } U \cup f \text{ was blue}$$

and

$$\begin{gathered}\text{in the recol., all col. in } e \setminus (U \cup f) \text{ remains the same} \\ \text{and all col. in } U \cup (e \cap f) \text{ are changed,}\end{gathered}$$

or the same with red/blue reversed, occurs.

Since

$$\Pr[B_{efU}] \leq 2^{1-2k+|e\cap f|} p^{|e\cap f|+|U|},$$

we have

$$\Pr[\cup_{efU} B_{efU}] \leq \sum_{e} \sum_{f:e\cap f\neq\emptyset} \sum_{U:U\subseteq e\setminus f} \Pr[B_{efU}].$$

It follows that

$$\begin{aligned}
\sum_{U:U\subseteq e\setminus f} \Pr[B_{efU}] &\leq \sum_{u=0}^{k-|e\cap f|} \binom{k-|e\cap f|}{u} \cdot 2^{1-2k+|e\cap f|} p^{|e\cap f|+u} \\
&= 2^{1-2k+|e\cap f|} p^{|e\cap f|} (1+p)^{k-|e\cap f|} \\
&\leq 2^{1-2k} e^{pk} \left(\frac{2p}{1+p}\right)^{|e\cap f|}.
\end{aligned}$$

Thus

$$\sum_{U:U\subseteq e\setminus f} \Pr[B_{efU}] \le 2^{1-2k}e^{pk}\left(\frac{2p}{1+p}\right)^{|e\cap f|} \le 2^{2-2k}e^{pk}p$$

yields

$$\begin{aligned}\Pr[\cup_e B_e] \le \Pr[\cup_{efU} B_{efU}] &\le \sum_e \sum_{f:e\cap f\neq\emptyset} 2^{2-2k}pe^{pk}\\ &\le |H|^2 2^{2-2k}pe^{pk}\\ &\le t^2pe^{pk},\end{aligned}$$

and hence

$$\begin{aligned}\Pr[\cup_e A_e \cup \cup_{e,f,U} B_{efu}] &\le \Pr[\cup_e A_e] + \Pr[\cup_{e,f,U} B_{efu}]\\ &\le 2te^{-pk} + t^2pe^{pk} < 1. \qquad \square\end{aligned}$$

3.2 Independence numbers of sparse graphs

Theorem (Ajtai, Komlós and Szemerédi ('81))
For a triangle-free graph G,

$$E[|I|] \geq \frac{cn \log t}{t}$$

($t = t(G)$: the average degree).

Ex. This implies that

$$R(3, t) \leq \frac{c' t^2}{\log t}.$$

For a graph G, let $\mathcal{I}$ denote the set of all independent sets of G. Take an independent set I in $\mathcal{I}$ uniformly at random so that

$$\Pr[I = J] = \frac{1}{|\mathcal{I}|} \quad \forall J \in \mathcal{I}.$$

I : (uniformly) random independent set

Theorem (Shearer) For a K_r-free graph G,

$$E[|I|] \geq \frac{c_r n \log t}{t \log \log t} \ .$$

Proof of AKS Theorem. (Alon's version of Shearer's proof)
Assume $maxdeg(G) \leq 2t$ (WHY?). For $v \in V(G)$, let

$$X_v = \begin{cases} t & \text{if } v \in I \\ |N(v) \cap I| & \text{otherwise,} \end{cases}$$

or equivalently,

$$X_v := t1(v \in I) + \sum_{w \sim v} 1(w \in I) \ .$$

Note that

$$\begin{aligned} \sum_{v \in V(G)} X_v &= t|I| + \sum_{v \in V(G)} \sum_{w \sim v} 1(w \in I) \\ &\leq t|I| + 2t|I| = 3t|I| \end{aligned}$$

Enough to show that

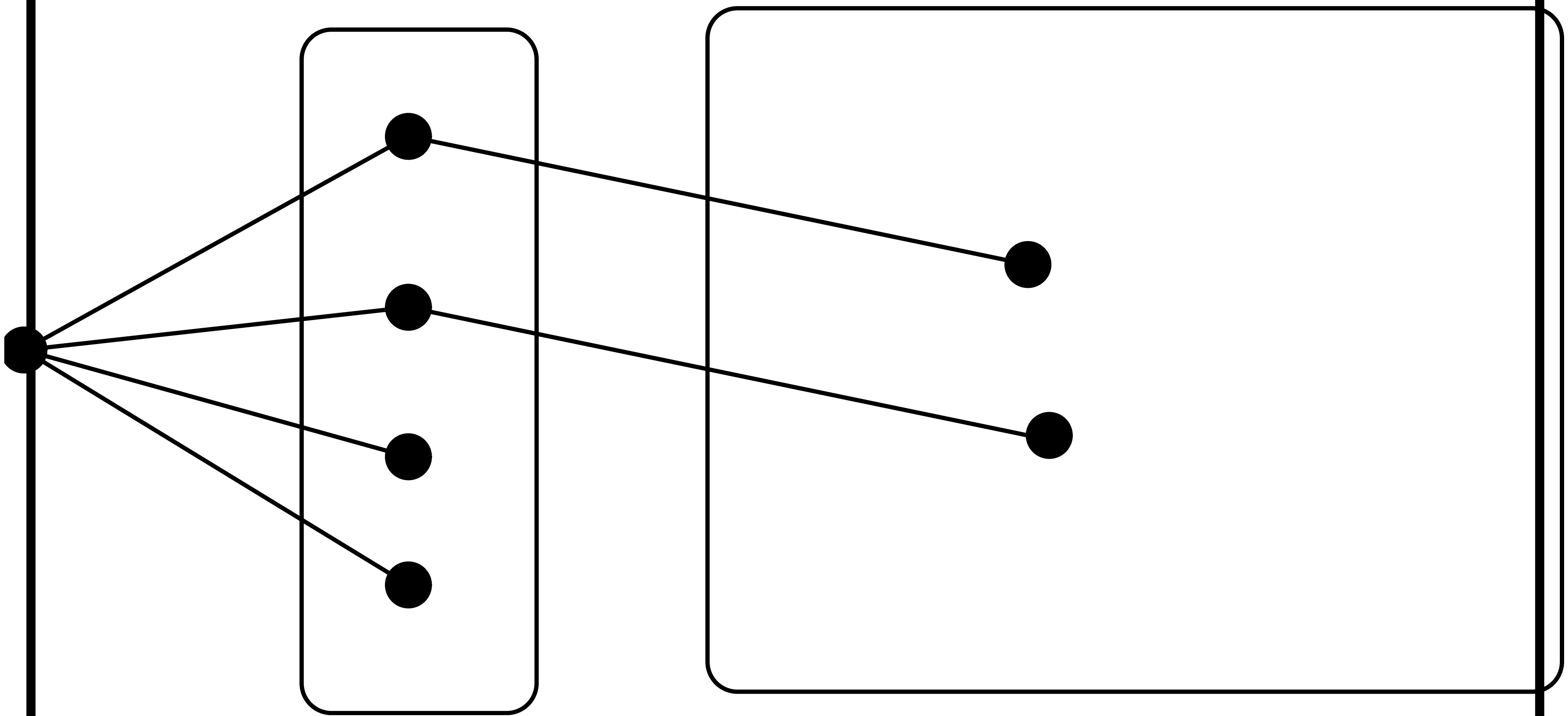
$$E[X_v] \geq c \log t,$$

for this implies

$$3tE[|I|] \geq E[\sum_v X_v] \geq cn \log t.$$

Let $J = I \setminus (\{v\} \cup N(v))$, we will show that

$$E[X_v|J] \geq c \log t \ .$$



$$E[X_v|J] = \frac{t}{1+2^r} + \frac{(r/2)2^r}{1+2^r} \ .$$

If $2^r \leq t/\log t - 1$, then

$$\frac{t}{1+2^r} \geq \log t \ .$$

If $2^r \geq t/\log t - 1$, then

$$\frac{(r/2)2^r}{1+2^r} \sim r/2 \geq \log t - \log\log t \ .$$

$\square$