

4.8 Perfect matchings in random uniform hypergraphs

Random d -regular graph on $V = \{1, \dots, n\}$

- Uniform Model $G^d(n)$
- Configuration Model $G_P^d(n)$
- Uniform Model $G^d(n)$
- Pairing Model $G_P^d(n)$
- Superposition Model (with/without multiple edges)
- Permutation Model $P^{2d}(n)$
- Degree-restricted process (Ruciński & Wormald)
- Star processes (Robalewska & Wormald)

Uniform Model $G^d(n)$

Each d -regular graph on V vertices is
equally likely to be chosen

- Random 1-regular graph

Random PM

- Random 2-regular graph G_2 :

Union of (disjoint) cycles

FACTS:

$\Pr[G_2 \text{ contains a PM}] = O(n^{-1/4})$

$$\Pr[G_2 \text{ contains a HC}] \sim \frac{e^{4/3} \pi^{1/2}}{2n^{1/2}} \approx \frac{1.876}{n^{1/2}}$$

Random 3-regular graph G_3 ??????

Configuration Model G_P^n

(Due to Bollobás ('79) similar models used by Bender and Canfield ('78), Békéssy, Békéssy and Komlós ('74), Wormald ('78))

For each vertex $v \in V$, consider d copies, or clones, $(v, 1), \dots, (v, d)$ of v . Let V^* be the set of all clones, i.e.,

$$V^* = \{(v, i) : v \in V, i = 1, \dots, d\} = V \times \{1, \dots, d\}.$$

Take a perfect matching on all clones. By contracting the clones of v into v , one may obtain a (multi) graph G^* . The uniform model may be realized from G^* by conditioning the event 'Simple' that neither loop nor multiple edge exists. That is,

$$\Pr[G^* = G | \text{Simple}]$$

are the same for all simple d -regular graphs on V . (WHY? Ex.)

$$\Pr[\text{Simple}] = \exp \left(-\frac{d^2 - 1}{4} - \frac{d^3}{12n} + O\left(\frac{d^2}{n}\right) \right)$$

(McKay & Wormald '91) For $d = o(\sqrt{n})$,

$$\Pr[\text{Simple}] \sim \exp \left(-\frac{d^2 - 1}{4} \right)$$

(McKay '85) For $d = o(d^{1/3})$,

$$\Pr[\text{Simple}] \sim \exp \left(-\frac{d^2 - 1}{4} \right)$$

(Bender & Canfield '78) For fixed d ,

• Random d -regular k -uniform hypergraph

For each vertex $v \in V$, consider d copies, or clones, $(v, 1), \dots, (v, d)$ of v . Let V^* be the set of all clones, i.e.,

$$V^* = \{(v, i) : v \in V, i = 1, \dots, d\} = V \times \{1, \dots, d\}.$$

Take a perfect matching on all clones. By contracting the clones of v into v , one may obtain a (multi) hypergraph H^* . The uniform model may be realized from H^* by conditioning the event 'Simple' that neither loop nor multiple edge exists. That is,

$$\Pr[H^* = H | \text{Simple}]$$

are the same for all simple d -regular graphs on V .

Ex. Is

$$\Pr[\text{Simple}] = e^{-\frac{2(k-1)}{(1+o(1))d}},$$

for $d = o(n^{k/2-1})$?

• Random d -regular k -partite k -uniform hypergraph

Let $H_k(n, d; k)$ be (uniform) random d -regular k -partite k -uniform hypergraph and $H_*^k(n, d; k)$ be the hypergraph (multi) hypergraph generated by the corresponding configuration model:

For each vertex $v \in V = V_1 \times \dots \times V_k$, $|V_i| = n$, consider d copies, or clones, $(v, 1), \dots, (v, d)$ of v . Let V^* be the set of all clones, i.e.,

$$V^* = \{(v, i) : v \in V, i = 1, \dots, d\} = V \times \{1, \dots, d\}.$$

Take a k -partite perfect matching on all clones. By contracting the clones of v into v , one may obtain a (multi) hypergraph $H_*^k = H_*^k(n, d; k)$. The uniform model may be realized from H_*^k by conditioning the event 'Simple' that neither loop nor multiple edge exists. That is,

$$\Pr[H_*^k = H | \text{Simple}]$$

are the same for all simple d -regular graphs on V .

FACT (Ex.): For $d = o(n^{k/2-1})$,

$$\Pr[\text{Simple}] = 1 - O(d^2 n^{-k+2}).$$

Theorem 4.3 For $1 \ll d = o(n^{k/2-1})$,

$$\Pr[H_k(n, d; k) \text{ contains PM}] = 1 - o(1).$$

Proof. We show this for (non-conditioned) $H_k^*(n, d; k)$. We also consider PM's consisting of ordered edges. To generate a random ordered PM on the set V^* , take an ordered dn/k -tuples of edges consisting of unlabelled (but ordered) k vertices. Then there are $((dn)!)^k$ ways to label vertices in the edges so that the set of the edges with labelled vertices is an ordered PM on V^* . The random ordered PM M is obtained when the labelling is chosen uniformly at random among all $((dn)!)^k$ labellings.

A collection R of n edges of M is called a V -PM if it induces a PM in V when clones of v is replaced by v . That is, for each $v \in V$, there is exactly one edge of M containing a clone of v . Clearly,

$$\Pr[R \text{ is a } V\text{-PM}] = \binom{(n!d^n)((d-1)(n)!)^k}{(dn)^n} = d^{kn}.$$

Let X be the number of V -PM in M , i.e.,

$$X = \sum_{R \text{ is a } V\text{-PM}} 1.$$

As there are $\binom{dn}{n}$ such R 's,

$$E[X] = \binom{dn}{n} d^{kn}.$$

For $p = \frac{1}{d}$,

$$\frac{E[X_2]}{E[X_2^2]} = \sum_{n=-k+1}^{\ell} \binom{(d-1)n}{n} \frac{\binom{pn}{n}}{\binom{\ell}{n}} \binom{\ell}{n} \left(\frac{p}{1}\right)^{\ell} \left(\frac{p}{1} - 1\right)^{n-\ell} \binom{\ell}{n} = \sum_{n=-k+1}^{\ell} \binom{(d-1)n}{n} \frac{\binom{pn}{n}}{\binom{\ell}{n}} \binom{\ell}{n} d^{\ell} (d-1)^{n-\ell} \binom{\ell}{n}.$$

Since

$$\frac{\binom{\ell}{n} \binom{pn}{n}}{\binom{(d-1)n}{n} \binom{\ell}{n}} = \frac{\binom{pn}{n}}{\binom{(d-1)n}{n} \binom{\ell}{n}} = \frac{\binom{pn}{n} (d-1)^{n-1} d^n}{\binom{\ell}{n} (d-1)^{n-1} d^n} = \frac{\Pr[\sum_{i=1}^n X_i = n]}{\Pr[\sum_{i=1}^n X_i = n+1]}.$$

where X_i 's are i.i.d Bernoulli random variables with mean p .

Hence,

$$\frac{E[X_2]}{E[X_2^2]} = \sum_n^{\ell=0} \left(\frac{\Pr[\sum_{i=1}^{\ell} X_i = n]}{\Pr[\sum_{i=1}^{\ell} X_i = n - \ell]} \right) \Pr \left[\sum_n^i X_i = \ell \right].$$

Let $\ell = pn + i, i = -pn, -pn + 1, \dots, (1 - p)n$, and

$$F_{\ell} = \left(\frac{\Pr[\sum_{i=1}^{\ell} X_i = n]}{\Pr[\sum_{i=1}^{\ell} X_i = n - \ell]} \right) \Pr \left[\sum_n^i X_i = \ell \right].$$

We will show that

1. If $|i| \leq r(pn)^{1/2}$ for a (large) constant r , then

$$\Pr \left[\sum_{i=1}^{\ell} X_i = pn + i \right] = (1 + o(1)) \Pr \left[\sum_{i=1}^{\ell} X_i = pn \right]$$

$$\sum_{|i| \leq r(pn)^{1/2}} F_{pn+i} = O \left(e^{-(1+o(1))r^2/2} \right).$$

2.

$$\Pr \left[\sum_{i=1}^{(d-1)n} X_i = n - \ell \right] \geq \Pr[X_1 = \dots = X_{\alpha n} = 1]$$

and $X^{\alpha n+1} = \dots = X^{(d-1)n} = 0$

$$= d^{\alpha n} (1 - d)^{(d-1-\alpha)n} \geq d^{0.1 \frac{n}{2}} e^{-n},$$

Proof of 4. If $\ell = (1 - \alpha)n$ with $\alpha \leq \delta := 0.1/k$, then

$$\sum_{i=1}^{(1-d)n} F^{dn+i} \leq d^{0.7n}$$

4.

$$\sum_{i=1}^{(1-d)n} F^{dn+i} = O\left(\delta^{-(k-1)/2} e^{-\Omega(\delta^{k-1} r_2)}\right).$$

3. For $\delta = 0.1/k$,

and

$$\Pr \left[\sum_n^{i=1} X_i = \ell \right] \leq \binom{\alpha n}{n} d^{(1-\alpha)n} \leq \left(\frac{\alpha}{e} \right)^{\alpha n} d^{(1-\alpha)n} \leq (10ke)^{0.1n} d^{0.9n}$$

gives

$$\left(\frac{\Pr \left[\sum_{i=1}^i X_i = n \right]}{\Pr \left[\sum_{i=1}^i X_i = n \right]} \right)^{k-1} \Pr \left[\sum_n^{i=1} X_i = \ell \right] \leq d^{0.8n}.$$

Thus,

$$\sum_n^{\ell = (1 - \frac{k}{0.1})n} \left(\frac{\Pr \left[\sum_{i=1}^i X_i = n \right]}{\Pr \left[\sum_{i=1}^i X_i = n \right]} \right)^{k-1} \Pr \left[\sum_n^{i=1} X_i = \ell \right] \leq n d^{0.8n} \leq d^{0.7n}$$

In the cases of 1, 2 and 3, we consider the ratio $\frac{F_\ell}{F_{\ell+1}}$. Recall

$$F_\ell = \left(\frac{\Pr \left[\sum_{i=1}^i X_i = n \right]}{\Pr \left[\sum_{i=1}^i X_i = n \right]} \right)^{k-1} \Pr \left[\sum_n^{i=1} X_i = \ell \right].$$

$$\left(\frac{u_z(d-1)}{(1+i)d} + 1 \right) \frac{\frac{u(d-1)}{i} - 1}{1} = \frac{(\frac{u(d-1)}{i} - 1)u_z(d-1)}{(1+i)d + u(d-2-d)}$$

$$= \frac{(j-u)(d-1)}{(1+j+u(2-p))d}$$

and

$$\left(\frac{u(d-1)}{i} - 1 \right) \frac{\frac{ud}{1+i} + 1}{1} = \frac{(\frac{ud}{1+i} + 1)ud(d-1)}{d(n-ud)} = \frac{(1+j)(d-1)}{d(n-u)}$$

For $\ell = pn + i, i = -pn, -pn + 1, \dots, (1-p)n - 1,$

$$\frac{\Pr[\sum_{i=1}^{d-1} X_i = n-1]}{\Pr[\sum_{i=1}^{d-1} X_i = n-\ell]} = \frac{(1-p)(n-\ell)}{d(2n+\ell+1)}$$

and

$$\frac{\Pr[\sum_{i=1}^n X_i = \ell]}{\Pr[\sum_{i=1}^n X_i = \ell+1]} = \frac{d(n-\ell)}{(1-p)(d-\ell+1)},$$

Notice that

1. If $|i| \leq r(pn)^{1/2}$ for a (large) constant r , then

$$1 - \frac{1 - \frac{n(d-1)}{i}}{1 + \frac{p(i+1)}{d-1}} = 1 + O(r(d/n)^{1/2}),$$

and

$$\Pr \left[\sum_{n^{(d-1)}}^{i=1} X_i = pn + i \right] = (1 + O(r(d/n)^{1/2}))^i \Pr \left[\sum_{n^{(d-1)}}^{i=1} X_i = pn \right]$$

$$= (1 + O(r(d/n)^{1/2}))^i \Pr \left[\sum_{n^{(d-1)}}^{i=1} X_i = pn \right]$$

2. For $i = -1, -2, \dots$,

$$F_{pn} = \frac{F_{pn}}{F_{pn-1}} \cdots \frac{F_{pn+i+1}}{F_{pn+i}} F_{pn+i} = \prod_{i=1}^{j=-1} \left(1 + \frac{ud}{j+1} \right) O\left(\frac{u}{j}\right)^{-1}$$

OR

$$F^{pn+i} = \prod_{|j|=-1}^{j=0} F^{pn} \left(1 - \frac{pn}{j} + O\left(\frac{n}{j}\right) \right) \leq e^{-\frac{i^2}{2}(1+o(1))} \frac{z^{pn}}{2}.$$

For the sum of F^{pn+i} over $i \leq -r(pn)^{1/2}$,

$$\sum_{i \leq -r(pn)^{1/2}} F^{pn+i} = O\left((pn)^{1/2} e^{-(1+o(1))r^2} F^{pn}\right) = O\left(e^{-(1+o(1))r^2/2}\right).$$

Similarly, for $i = 1, \dots, (1-p)n - \delta n$ with a constant $\delta = 0.1/k$,

$$F^{pn+i} \leq F^{pn} \prod_i^{j=1} \left(1 + \frac{z^{pn}}{\delta^{k-1} j} + O\left(\frac{n}{j}\right) \right)^{-1}$$

$$\leq (1 + \delta^{k-1} / \mathfrak{E})^{-\max\{i-pn, 0\}} \exp\left(-\Omega\left(\frac{pn}{\delta^{k-1} \min\{i, pn\}^2}\right)\right)$$

$$\left(O \left(\delta^{-(k-1)/2} e^{-\Omega(\delta^{1-k})} \right) \right) =$$

$$\left({}^{ud}F \left(\delta^{-(k-1)/2} e^{-\Omega(\delta^{1-k})} \right) \right) = \sum_{u \in \mathcal{U}(d-1)} {}^{i+ud}F \left(\delta^{-(k-1)/2} e^{-\Omega(\delta^{1-k})} \right)$$

and

4.9 Covering hypercube

Recall

n -cube \mathcal{Q}_n :

$$\{-1, 1\}^n = \{x_i : x_i = 1 \text{ or } -1, i = 1, \dots, n\}$$

Let X_1, \dots, X_m be (mutually) independent uniform random vectors in \mathcal{Q}_n , in particular,

$$\Pr[X_j = u] = 2^{-n} \text{ for any } u \in \mathcal{Q}_n.$$

Ex. Let $Y_w, w \in \mathcal{Q}_n$, be the indicator random variable for the event $w \cdot X_i > 0$ for all $i = 1, \dots, m = 0.5n$, and $Y = \sum_{w \in \mathcal{Q}_n} Y_w$. Show that

$$\frac{E[Y_2]}{E[Y]^2} \rightarrow \infty.$$

5 Chernoff Bound

E.g. Degrees of $G = G(n, 1/2)$

Recall

Each of $\binom{n}{2}$ edges is in G
probability $1/2$

For fixed v ,

$$E[d(v)] = (n - 1)/2.$$

How about

$$\Pr[|d(v) - n/2| \geq n^{3/4}] = ??$$

(Second Moment Method) For $w \neq v$, let

$$X_w = \begin{cases} 1 & \text{if the edge } \{v, w\} \text{ in } G \\ 0 & \text{otherwise} \end{cases},$$

and

$$X = \sum_{w \neq v} X_w = d(v).$$

Expectation:

$$E[X] = \sum_{w \neq v} E[X_w] = \frac{n-1}{2}.$$

Variance: Since

$$\text{cov}(X_w, X_u) = E[X_w X_u] - E[X_w]E[X_u] = 0,$$

$$\sigma_2(X) = \sum_{i=1}^w \sigma_2(X_i) = \frac{w}{n-1}.$$

Thus Chebyshev's inequality gives

$$\Pr[|d(v) - n/2| \geq n^{3/4}] \leq \frac{1}{(2n^{1/4})^2} = \frac{1}{4n^{1/2}}.$$

How about

$$\Pr[\exists v, |d(v) - n/2| \geq n^{3/4}] = ??$$

The second moment method gives

$$\Pr[\exists v, |d(v) - n/2| \geq n^{3/4}] \leq n \cdot \frac{1}{4n^{1/2}} = n^{1/2}/4 \quad ???$$

CAN WE DO BETTER??

5.1 Chernoff Bound

Let

$$X = \sum_{i=1}^n X_i,$$

where X_i : i.i.d. with

$$\Pr[X_1 = 0] = \Pr[X_1 = 1] = 1/2.$$

Then

$$\Pr[|X - n/2| \geq \lambda] \leq 2e^{-2\lambda^2/n}.$$

Proof. For any $\alpha > 0$,

$$\begin{aligned} \Pr[X - E[X] \geq \lambda] &\leq \Pr[e^{\alpha(X - E[X])} \geq e^{\alpha\lambda}] \\ &\leq E[e^{\alpha(X - E[X])}] e^{-\alpha\lambda}. \end{aligned}$$

Thus

$$\begin{aligned} E[e^{\alpha(X-1/2)}] &= \prod_{i=1}^n E[e^{\alpha X_i}] \\ &= \prod_{i=1}^n (\cosh(\alpha/2))^n \\ &\leq e^{\alpha^2 n/8}, \end{aligned}$$

yields

$$\Pr[X - E[X] \geq \lambda] \leq \exp(-\alpha\lambda + \alpha^2 n/8)$$

(for all $\alpha > 0$). Taking $\alpha = 4\lambda/n$, we have

$$\Pr[X - E[X] \geq \lambda] \leq \exp(-2\lambda^2/n).$$

And similarly,

$$\Pr[X - E[X] \leq -\lambda] \leq \exp(-2\lambda^2/n).$$

□

A general version: For

$$X = \sum_{i=1}^n X_i,$$

where X_i are independent with

$$\Pr[X_1 = 0] = 1 - p_i, \Pr[X_1 = 1] = p_i,$$

and $p = \frac{1}{n} \sum_i p_i,$

$$\Pr[X - E[X] \leq -\lambda] \leq e^{-\lambda^2/(2pn)},$$

and, for all $a > 0,$

$$\Pr[X - E[X] \geq \lambda] \leq e^{-(a+pn) \ln(1+a/pn)}.$$

In particular,

$$\Pr[X - E[X] \geq \lambda] \leq e^{-\lambda^2/(2pn) + \lambda^3/(2(pn)^2)}.$$

Examples

- A football team A win each game with probability $1/3$. Outcomes of games are indep. Upper bound for the probability that A win more than $n/2$ games among n games.

$$\Pr[X > n/2] = \Pr[X > (1+1/2)n/3] < e^{-(n/3)(1/12)} < (0.965)^n.$$

- (Set Balancing) $A : n \times n$ matrix. Find $b \in \{-1, 1\}^n$ minimizing $\|Ab\|_\infty$.

Choose b s.t. each entry of b is selected from $\{-1, 1\}$

independently and equiprobably. For any row r of A , by

Chernoff's bound,

$$\Pr[\text{inner prod. of } r \text{ and } b] > 2\sqrt{n \ln n}] < 2e^{-4n \ln n / 2n} = 2/n^2.$$

$$\Rightarrow \Pr[\|Ab\|_\infty > 2\sqrt{n \ln n}] < 2/n.$$

- Degrees of $G(n, 1/2)$

$$\Pr[d(v) - n/2 \geq n^{3/4}] \leq e^{-2n^{3/2}/n} = e^{-2n^{1/2}}$$

and

$$\Pr[\exists v, |d(v) - n/2| \geq n^{3/4}] \leq ne^{-2n^{1/2}} \rightarrow 0.$$

Actually,

$$\Pr[d(v) - n/2 \geq \sqrt{n \ln n}] \leq e^{-2n \ln n/n} = n^{-2},$$

and

$$\Pr[|d(v) - n/2| \geq \sqrt{n \ln n}] \leq n \cdot n^{-2} = 1/n \rightarrow 0.$$

Generally, for $G(n, p)$,

$$\Pr\{|d(v) - pn| \geq 2\sqrt{pn \ln n}\} \leq ???$$

5.2 Edge discrepancy

τ : 2-coloring of all edges of K_n , say Red and Blue.

Let the (edge) discrepancy of τ denote

$$h(n, \tau) = \max_{S \subseteq [n]} |\# \text{ of Red edges in } S - \# \text{ of Blue edges in } S|,$$

and

$$h(n) = \min_{\tau: 2\text{-coloring}} h(n, \tau).$$

Theorem 5.1

$$h(n) \leq cn^{3/2}.$$

Proof. Randomly 2-color all edges. For each edge e , let

$$X_e = \begin{cases} 1 & \text{if } e \text{ is Red} \\ -1 & \text{if } e \text{ is Blue} \end{cases}.$$

Then for all subset S ,

$$X_S = |\#(RED) \text{ in } S - \#(BLUE) \text{ in } S| = \sum_{e \in S} X_e.$$

Note that

$$E[X_e] = 0, \text{ and } E[X_S] = 0.$$

Thus

$$\Pr[|X_S| \geq m] \leq \exp\left(-\frac{2 \binom{2}{|S|}}{m^2}\right) \leq \exp\left(-\frac{m^2}{n^2}\right),$$

and

$$\Pr[\exists S, |X_S| \geq m] \leq 2^n \cdot \exp\left(-\frac{m^2}{n^2}\right) > 1$$

by taking $m = (\ln 2)^{1/2} n^{3/2} + \sqrt{n/2}$.

□

5.3 Discrepancy

H : hypergraph on n vertices

χ : 2-coloring of all vertices, say Red and Blue

As usual,

$$X_v = \begin{cases} 1 & \text{if } v \text{ is Red} \\ -1 & \text{if } v \text{ is Blue} \end{cases}.$$

For each edge A , discrepancy $\text{disc}(A)$ of A is the absolute value of

$$X_A = \sum_{v \in A} X_v = |\# \text{ of (RED) in } A - \# \text{ of (BLUE) in } A|,$$

and discrepancy $\text{disc}(H, \chi)$ of H w.r.t. χ is

$$\text{disc}(H, \chi) = \max_{A \in H} |X_A| \text{ and } \text{disc}(H) = \min_{\chi} \text{disc}(H, \chi).$$

Theorem If

$$|H| = n,$$

then

$$\text{disc}(H) \leq \sqrt{2n \ln(2n)}.$$

Proof.

χ : random 2-coloring of all vertices

For each $A \in H$,

$$\Pr[\text{disc}(A) > \lambda] = \Pr[|X_A| > \lambda] > 2 \exp \left(- \frac{\lambda^2}{2|A|} \right) \leq 2 \exp \left(- \frac{\lambda^2}{2n} \right),$$

and

$$\Pr[\exists A \in H, \text{disc}(A) \geq \lambda] > n \cdot 2 \exp \left(- \frac{\lambda^2}{2n} \right) = 1,$$

by taking $\lambda = \sqrt{2n \ln(2n)}$.

5.4 Random graph $G = G(n, p)$

Let $p = 0.2n^{-1/2}$, $t = c\sqrt{n \ln n}$ and $\beta = 0.12e^2 = 0.8866 \dots > 1$.
 Then for a subset T of the vertex set with $|T| = t$,

$$\Pr \left[e(T) > \beta p \binom{t}{2} \right] \leq \exp \left(- \frac{\binom{t}{2} d_{\beta}^{(2)}}{\binom{t}{2}} \right) = \exp \left(- \frac{t}{\binom{t}{2}} \right)$$

and (by Boole's inequality)

$$\Pr \left[\exists T, e(T) > \beta p \binom{t}{2} \right] \leq \binom{n}{t} \exp \left(- \frac{t}{\binom{t}{2}} \right) \leq \exp \left(- \frac{t}{\binom{t}{2}} + t \ln n \right) = o(1)$$

for $p = 0.2n^{-1/2}$, $t = c\sqrt{n \ln n}$ and $\beta = 0.12e^2 > 1$ with large c .

Theorem (Erdős '61)

$$R(3, t) \geq C \left(\frac{t}{\ln t} \right)^2.$$

Proof. (Krivelevich) It is enough to show that

\exists triangle-free graph G^* on n vertices with

$$\alpha(G^*) \leq c\sqrt{n \log n}.$$

From the random graph $G(n, p)$ with $p = 0.2n^{-1/2}$, take a “maximal” (under \subseteq) family \mathcal{F} of edge disjoint triangles.

Deleting all edges belongs to triangles in \mathcal{F}

we obtain G^* on n vertices.

Clearly,

G^* is triangle-free.

For each subset T of size $t = c\sqrt{n \log n}$, let

$f(T)$ denote the maximum number of edge disjoint

triangles with at least one edge in T .

Then

we deleted at most $3f(T)$ edges from T .

If, in $G = G(n, p)$ with $p = 0.2n^{-1/2}$,

$e(T) > 3f(T)$ for all T ,

T still has at least one edge. (So, T is not an independent set)

For $\beta = 0.12e^2 < 1$, it is enough to show that

$$\Pr \left[\exists T, e(T) > \beta p \binom{2}{t} \right] = o(1)$$

(recall $p = 0.2n^{-1/2}$, $t = c\sqrt{n \ln n}$), and

$$\Pr \left[\exists T, f(T) \geq (\beta p/3) \binom{2}{t} \right] = o(1).$$

To prove

$$\Pr \left[\exists T, f(T) \geq (\beta/3)p \binom{2}{t} \right] = o(1),$$

we consider

Lemma (Erdős-Tetali's disjointness lemma) Suppose \mathcal{A} be a collection of events with

$$\sum_{A \in \mathcal{A}} \Pr[A] \leq \mu.$$

Then, for any $l > 0$,

$$\sum_{\substack{\{A_1, \dots, A_l\} \subseteq \mathcal{A} \\ \text{mut. indep. events}}} \Pr[\bigcap_{i=1}^l A_i] \leq \frac{\mu^l}{l!}.$$

Proof.

$$\begin{aligned}
 & \sum_{\{A_1, \dots, A_l\} \subseteq \mathcal{A}} \Pr[\bigcap_{i=1}^l A_i] \\
 & \qquad \qquad \qquad \text{mut. indep. events} \\
 & = \sum_{\substack{\{A_1, \dots, A_l\} \subseteq \mathcal{A} \\ \text{mut. indep. events}}} \prod_{i=1}^l \Pr[A_i]
 \end{aligned}$$

$$\begin{aligned}
 & \leq \sum_{\{A_1, \dots, A_l\} \subseteq \mathcal{A}} \prod_{i=1}^l \Pr[A_i]
 \end{aligned}$$

$$\begin{aligned}
 & = \sum_{(A_1, \dots, A_l) \subseteq \mathcal{A} \times \dots \times \mathcal{A}} (1/l!) \prod_{i=1}^l \Pr[A_i]
 \end{aligned}$$

$$\begin{aligned}
 & = \sum_{A_l \in \mathcal{A}} (1/l!) \sum_{A_1 \in \mathcal{A}} \dots \sum_{A_l \in \mathcal{A}} \Pr[A_l]
 \end{aligned}$$

$$\leq \frac{\mu_l}{l!}$$

Proof of

$$\Pr[\exists T, f(T) \geq (\beta/3)p \binom{2}{t}] = o(1).$$

Recall $G = G(n, p)$ with $p = 0.2n^{-1/2}/3$ and $\beta = 0.12e^2 < 1$.

For a subset T of size $t = c\sqrt{n \ln n}$ and each triangle efg in K_n ,

define

$$A_{efg} = \{efg \in G\},$$

and

$$A = \{A_{efg} : e \subseteq T\}.$$

Then

$$\sum_{A_{efg} \in A} \Pr[A_{efg}] \leq n \binom{2}{t} p^3 = 0.04p \binom{2}{t} =: \mu.$$

For $l = (\beta p/3) \binom{2}{t}$,

$$\Pr[f(T) \geq l] = \Pr \left[\bigcup_{i=1}^l A_i \text{ occurs for some MI events } A_1, \dots, A_l \in \mathcal{A} \right]$$

$$\leq \sum_{\{A_1, \dots, A_l\} \subseteq \mathcal{A} \text{ MI events}} \Pr[\bigcap_{i=1}^l A_i] \leq \frac{l!}{l!} \leq \left(\frac{l}{e}\right)^l.$$

Since

$$\left(\frac{l}{e}\right)^l = \exp \left(l \ln(0.12e/\beta) \binom{2}{t} \right) = \exp \left(-0.04e^{1/2} n^{-1/2} \binom{2}{t} \right),$$

$$\beta = 0.12e^2, l = (\beta p/3) \binom{2}{t}, \text{ and } n = 0.04p \binom{2}{t},$$

$$(t = c\sqrt{n \ln n}).$$

$$\begin{aligned} & \leq \Pr[\exists T, f(T) \geq t] \leq \binom{n}{t} \exp\left(-0.04e_2 n^{-1/2} \binom{2}{t}\right) \\ & = o(1) \quad \text{if } c > 50/e_2 \end{aligned}$$

5.5 Routing in a Parallel Computer

- A model for a network of parallel processors
 - A directed graph on N nodes (or processors).
 - Each edge represent communication link between processors.
 - Synchronous message passing.
 - Each link send at most one message (*packet*) per step.
 - Each node contains a queue.
- **Permutation Routing Problem**
 - Each node i has a packet p_i destined for a node $\delta(i)$, where δ is a permutation of all nodes.
 - **Oblivious Algorithms**
 - The route followed by p_i depends on $\delta(i)$ alone.

Theorem 5.2 For any deterministic oblivious permutation routing algorithm on a network of N nodes each of out-degree d , there is an instance of permutation routing requiring $\Omega(\sqrt{N/d})$ steps.

* Reason: some edge may have lots of paths through it.

An Example:

- For d -dimensional hypercube, there are $N = 2^d$ nodes and dN directed edges. Consider the bit fixing routing strategy scanning from left to right. For example, If $i = (1011)$ and $\delta(i) = (0000)$, then

$$(1011) \leftarrow (0011) \leftarrow (0001) \leftarrow (0000).$$

- There can be a paths of length d that is a lower bound on routing time.

Homework: Consider the *transpose* permutation: writing i as the concatenation of two $d/2$ -bit strings a_i and b_i , we want to route $a_i b_i$ to $b_i a_i$. Show the bit fixing strategy takes $\Omega(\sqrt{N/d})$ steps on this permutation.

Randomized routing algorithm:

Phase 1: For any node i , pick a random intermediate destination

$\sigma(i)$ from $\{1, 2, \dots, N\}$. Packet p_i travels to the node $\sigma(i)$. (σ is

not necessarily permutation)

Phase 2: Packet p_i travels from $\sigma(i)$ to the destination node $\delta(i)$.

We use the bit-fixing strategy to determine routes.

Lemma 5.3 *Once two routes separate, they do not rejoin.*

Proof. It follows as a bit checked once would not be checked again.

□

Lemma 5.4 *Let the route of p_i follow the sequence of edges*

$p_i = (e_1, \dots, e_k)$. *Let S be the set of packets other than p_i whose routes pass through at least one of $\{e_1, \dots, e_k\}$. Then, the delay*

incurred by p_i is at most $|S|$.

Proof. Idea:

Match each delay of p_i with

some packet departing from p_i .

Let $e_j = (v_j, v_{j+1})$. If a packet $p \in S$ is at v_j at time t and ready to follow e_j , then the lag $\ell_t(p)$ of p at t is defined as $t - j$. For $\ell \geq 1$, let $t(\ell)$ be the last time when there is a packet $p \in S$ with lag ℓ . That is,

$$t(\ell) = \max\{t : \exists p \in S, \ell_t(p) = \ell\}.$$

Note that the lag $\ell_t(p)$ of p is the delay of p up to time t . If the lag of p becomes $\ell + 1$ from ℓ at a time t , then there is a $p \in S$ with lag ℓ at time t . This implies that $t(\ell)$ is well-defined. We may now take a packet $p \in S$ with lag ℓ at time $t(\ell)$. Hence, at time $t(\ell) + 1$, the packet p itself or another packet $p' \in S$ follows an edge e_j in p_i . This packet is ready to follow an edge other than e_{j+1} . Otherwise, its lag would be ℓ at time $t(\ell) + 1$. Therefore, we found a packet $p' \in S$ with lag ℓ just before it leaves p_i .

□

Let $X_{i,j} = 1$ if p_i and p_j share at least one edge, and 0 otherwise.

Then the total delay of p_i is at most $Y_i := \sum_{\substack{j=1 \\ j \neq i}}^N X_{i,j}$.

Conditioned on $p_i = (e_1, \dots, e_k)$, we count the expected number of p_j containing $e = e_l$, $l = 1, \dots, k$. Suppose e is an ordered pair of

$v = (v', x, v'')$ and $w = (w', y, w'')$ with $x \neq y$. Then p_j contains e if and only if the corresponding packet must start from a vertex of

the form $(*, x, v'')$ and its destination must be of the form $(v', y, *)$,

where $*$ can be replaced by any vectors of the appropriate length.

Assuming v' is of length a , the expected number of such p_i is

$$(2^a - 1)2^{-(a+1)} \leq 1/2. \text{ Thus,}$$

$$E \left[\sum_{\substack{j=1 \\ j \neq i}}^N X_{i,j} \mid p_i = (e_1, \dots, e_k) \right] \leq \sum_k \sum_{\substack{j=1 \\ j \neq i}}^N 1(e_l \text{ in } p_j) \mid p_i = (e_1, \dots, e_k)$$

$$\leq k/2 \leq d/2.$$

For fixed i and $Y_i = \sum_{\substack{j=1 \\ j \neq i}}^N X_{i,j}$, Chernoff bound gives

$$\Pr[Y_i \geq 6d | \sigma(i)] \leq \Pr[Y_i - E[Y_i] \geq 11d/2 | \sigma(i)] \leq e^{-11d/2 - (11d/2) \ln 12} \leq e^{-8d},$$

which yields

$$\Pr[Y_i \geq 6d] \leq e^{-8d},$$

and

$$\Pr[\exists i, Y_i \geq 6d] \leq 2^d e^{-8d} \leq e^{-7d}.$$

We have proved that

Theorem 5.5 *With probability at least $1 - e^{-7d}$, every packet reaches its intermediate destination in Phase 1 in $7d$ or fewer steps.*

Corollary 5.6 *With probability at least $1 - 2e^{-7d}$, every packet reaches its destination in $14d$ or fewer steps.*

5.6 A Wiring Problem

- Model

- A *gate array* consisting of $\sqrt{n} \times \sqrt{n}$ array of gates.
- Manhattan wiring.
- A set of *nets*, each of which is a set of gates to be connected by a wire.
- Two sequential phases: *global wiring* and *detailed wiring*
- Each boundary between gates has a limit on number of crossing wires.
- General problem: minimize max crossing number.
- Simplification: each net has *two gates* and, only *one-bend route* is allowed. In particular, there are only two choices for a route.

• Integer Programming (IP)

- For each net i , $x_i = 1$ means the route of net i goes horizontally first from the left-end vertex of net i .
- For a boundary b ,

$$S_b = \{i | \text{net } i \text{ through } b \text{ if } x_i = 1\}, \text{ and}$$

$$T_b = \{i | \text{net } i \text{ through } b \text{ if } x_i = 0\}.$$

– IP

$$\min w$$

subject to $x_i \in \{0, 1\}$

\forall nets i

$$\sum_{i \in S_b} x_i + \sum_{i \in T_b} (1 - x_i) \leq w$$

\forall boundaries b

- Let w_0 be the minimum of the IP.

• Relaxation

- IP is NP-hard but LP is in P .
- Relax $x_i, y_i \in \{0, 1\}$ to $x_i, y_i \in [0, 1]$.
- Linear Programming (LP)

min w

subject to $x_i \in [0, 1]$

\forall nets i

$$\sum_{i \in S_i} x_i + \sum_{i \in T_i} (1 - x_i) \leq w \quad \forall \text{ boundaries } b$$

- Suppose solutions are \hat{x}_i 's with the minimum value \hat{w} . Then $\hat{w} \leq w_0$.

- Rounding: $\bar{x}_i = 1$ if and only if $\hat{x}_i \geq 1/2$. Then

$$\sum_{i \in S_i} \bar{x}_i + \sum_{i \in T_i} (1 - \bar{x}_i) \leq \sum_{i \in S_i} 2\hat{x}_i + \sum_{i \in T_i} 2(1 - \hat{x}_i) \leq 2\hat{w} \leq 2w_0.$$

• Randomized rounding:

$$\Pr[X_i = 1] = \hat{x}_i \quad \text{and} \quad \Pr[X_i = 0] = 1 - \hat{x}_i,$$

independently of all other nets. Then, for each boundary b and

$$Y_b = \sum_{i \in S_b} X_i + \sum_{i \in T_b} (1 - X_i),$$

$$E[Y_b] = \sum_{i \in S_b} \hat{x}_i + \sum_{i \in T_b} (1 - \hat{x}_i) \leq \hat{w} \leq w_0.$$

We may define $Y_b^* \geq Y_b$ and $E[Y_b^*] = w_0$ to obtain

$$\Pr[Y_b \geq w_0 + (6w_0 \ln n)^{1/2}] \leq \Pr[Y_b^* \geq w_0 + (6w_0 \ln n)^{1/2}] \leq e^{-3 \ln n + (6w_0 \ln n)^{1/2}}.$$

If $w_0 \geq 100 \ln n$, then

$$\Pr[Y_b \geq w_0 + (6w_0 \ln n)^{1/2}] \leq e^{-2 \ln n} = n^{-2}$$

and

$$\Pr[\exists b, Y_b \geq w_0 + (6w_0 \ln n)^{1/2}] \leq 2/n.$$

6 Martingale

6.1 Conditional Expectations

$Y : \text{RV} \iff Y_2, Y_3 - Y - 2, f(Y)$ are RV's

for any function f .

Notice that if the value of Y is given we know (deterministically) the value of $f(Y)$.

For RV's X, Y , let

$$f_k(Y) = \text{Pr}[X = k|Y].$$

If $Y = l$, then

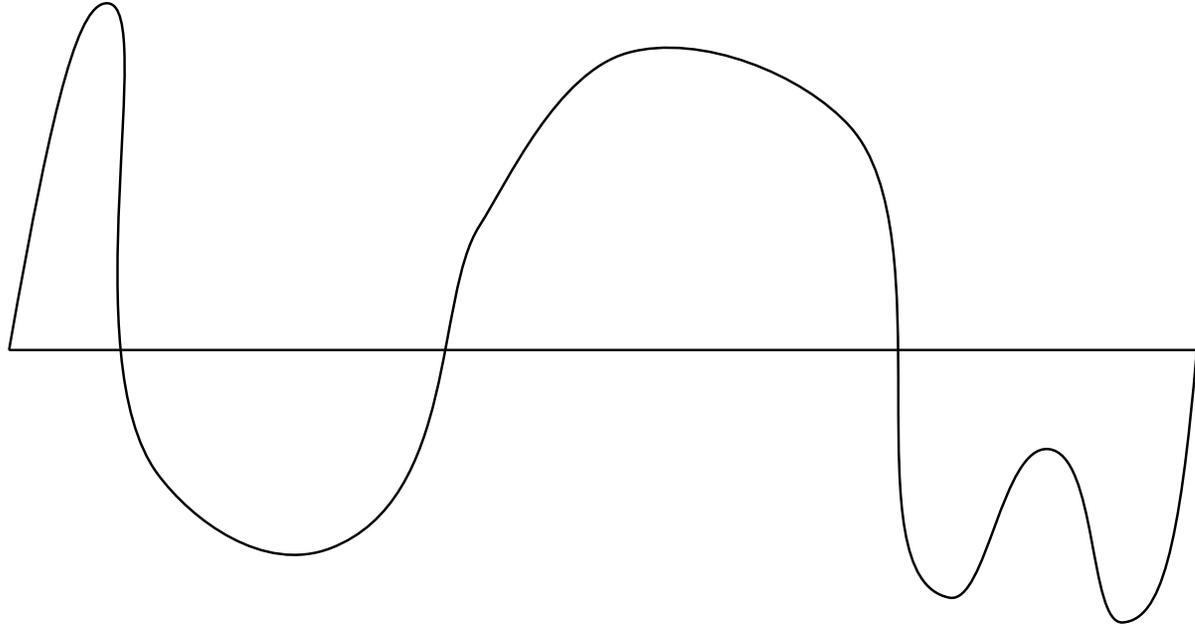
$$f_k(Y) = \text{Pr}[X = k|Y = l] = \frac{\text{Pr}[X = k, Y = l]}{\text{Pr}[Y = l]}.$$

The RV

$$g(Y) = \sum_{-\infty}^{\infty} k \Pr[X = k|Y]$$

is denoted by $E[X|Y]$.

Meaning: Average of X on $A_l := \{Y = l\}$.



E.g.

X_i : the number of times i appears in n independent trials of unbiased 6-sided die.

$$E[X_1 | X_2] = E[X_1 | X_2, X_3] =$$

E.g.

X_i : the number of times i appears in n independent trials of unbiased 6-sided die.

$$E[X_1|X_2] = (n - X_2)/5$$
$$E[X_1 | X_2, X_3] =$$

E.g.

X_i : the number of times i appears in n independent trials of unbiased 6-sided die.

$$E[X_1 | X_2] = (n - X_2) / 5$$

$$E[X_1 | X_2, X_3] = (n - X_2 - X_3) / 4$$

• Properties

1. If X, Y are independent then

$$E[X|Y] = E[X] \quad (\text{a constant}),$$

$$E[Y^2|Y] = Y^2, \quad E[f(Y)|Y] = f(Y).$$

Let τ_1, \dots, τ_m are i.i.d. RV's and $X = X(\tau_1, \dots, \tau_m)$.

E.g. In $G(n, p)$,

$$\Pr[\tau_e = 1] = p,$$

where $\tau_e = 1$ if and only if the edge e is in $G(n, p)$.

RV's : $\omega(G), \alpha(G), \chi(G), \dots$

For simplicity, let

the set of edges $K_n = \{e_1, \dots, e_{\binom{n}{2}}\}$ and $\tau_{e_i} = \tau_i$.

2. For RV's X, X', Y ,

$$E[X + X'|Y] = E[X|Y] + E[X'|Y], \quad E[E[X|Y]] = E[X].$$

For any function $f(y)$,

$$E[f(X|Y)] = E[f] \cdot E[X|Y].$$

6.2 Martingale

For RV $X = X(\tau_1, \dots, \tau_m)$, let

$$X_k = E[X | \tau_1, \dots, \tau_k]$$

(for any independent τ 's). Then

$$E[X_k | \tau_1, \dots, \tau_l] = \begin{cases} X_k & \text{if } l \geq k, \\ X_l & \text{if } l \leq k \end{cases}$$

or

$$E[X_k | \tau_1, \dots, \tau_l] = X_{k \wedge l}, \quad k \wedge l = \min\{k, l\}.$$

In particular,

$$E[X_k | \tau_1, \dots, \tau_{k-1}] = X_{k-1}.$$

Notice also that

$$X_m = E[X | \tau_1, \dots, \tau_m] = X.$$

Definition A sequence of RV's X_0, X_1, \dots, X_m is a *martingale sequence* if for all $i > 0$,

$$E[X_i | X_0, X_1, \dots, X_{i-1}] = X_{i-1}.$$

More generally, for a collection of random variables $\{T_i\}_{i=1}^m$, a sequence of RV's X_0, X_1, \dots, X_m is a *martingale sequence* with respect to $\{T_i\}_{i=1}^m$ if

$$E[X_i | T_0, T_1, \dots, T_{i-1}] = X_{i-1},$$

where $T_0 \equiv 0$.

E.g. 1. (Sum of i.i.d RV's) Let X_1, \dots, X_m be i.i.d with $E[X_1] = 0$, e.g. $X_1 = \pm 1$ w/ $\text{pr} = 1/2$. Then

$$\{S_k := X_1 + \dots + X_k\}$$

is a martingale.

2. (Polya's Urn Scheme)

- An urn with b black balls and w white balls.
- At each step, one randomly chosen ball is replaced by c balls of the same color. Let X_i be the fraction of black balls after the i^{th} trial.

- X_0, X_1, X_2, \dots : Martingale sequence.

3. (Edge exposure Martingale)

- Probability space: $\mathcal{G}_{n,p}$.
- Label $m = n(n-1)/2$ possible edges with $1, 2, \dots, m$.
- For $j = 1, 2, \dots, m$, I_j is indicate r.v. for edge labelled j .
- F : real valued ftn. defined over the space of all graphs.
- (Chromatic number, independent number, etc)
- $X_k = E[F(G) \mid I_1, I_2, \dots, I_k]$ and $X_0 = E[F(G)]$, $X_m = F(G)$.

6.2.1 Martingale Inequality

- Martingale difference sequence

For martingale $\{X_k\}_{k=0}^m$ with $X = X_m$,

mart. diff. seq. is $\{Y_k := X_k - X_{k-1}\}_{k=1}^m$.

Then

$$Y := \sum_{k=1}^m Y_k = X - E[X], \quad E[Y_k | \mathcal{T}_1, \dots, \mathcal{T}_{k-1}] = 0$$

Main Idea: If each Y_k is small in some sense

then we 'EXPECT' that $X - E[X]$ is small.

Recall $Y = X - E[X]$

Lemma If

$$E[e^{\alpha Y_k} | T_1, \dots, T_{k-1}] \leq e^{c_2^k \alpha^2 / 2}$$

for all $\alpha > 0$, then

(a) $E[e^{\alpha Y}] \leq \exp\left(\sum_k c_2^k \alpha^2 / 2\right)$

(b) $\Pr[X - E[X] \geq \lambda] \leq \exp\left(-\lambda^2 / 2 \sum_k c_2^k\right)$

for all real number $\lambda > 0$.

Proof. EASY (a) \iff (b): By Markov's inequality,

$$\begin{aligned} \Pr[Y \geq \lambda] &= \Pr[e^{\alpha Y} \geq e^{\alpha \lambda}] \\ &\leq e^{-\alpha \lambda} E[e^{\alpha Y}] \\ &\leq \exp\left(-\alpha \lambda + \alpha^2 \sum_{k=1}^k c_k^2 / 2\right) \end{aligned}$$

Taking $\alpha = \frac{\lambda}{\sum_{k=1}^k c_k^2}$, we obtain (b).

For (a), we show

$$E[e^{\alpha(Y_1 + \dots + Y_i)}] \leq \exp\left(\sum_{i=1}^k c_2^i \alpha^2 / 2\right)$$

for all $i = 1, \dots, n$ by induction. For $i = 1$,

$$E[e^{\alpha Y_1}] \leq \exp\left(c_1^1 \alpha^2 / 2\right).$$

For $i > 1$, the induction hypothesis gives

$$\begin{aligned} E[e^{\alpha(Y_1 + \dots + Y_i)}] &= E[E[e^{\alpha(Y_1 + \dots + Y_i)} | \tau_1, \dots, \tau_{i-1}]] \\ &= E[e^{\alpha(Y_1 + \dots + Y_{i-1})} E[e^{\alpha Y_i} | \tau_1, \dots, \tau_{i-1}]] \\ &\leq E[e^{\alpha(Y_1 + \dots + Y_{i-1})} e^{c_2^i \alpha^2 / 2}] \\ &\leq \exp\left(\sum_{i=1}^k c_2^i \alpha^2 / 2\right). \end{aligned}$$

□

Corollary (Azuma) If

$$|Y_k| \leq c_k$$

then

$$\Pr \left[|X - E[X]| \geq \lambda \right] \leq 2 \exp \left(- \frac{\lambda^2}{2 \sum_{k=1}^n c_k^2} \right)$$

for all $\lambda > 0$.

Proof. As the function $f(x) = e^x$ is convex, and

$$E[Y_k | T_1, \dots, T_{k-1}] = 0, \text{ we have}$$

$$E[e^{\alpha Y_k} | T_1, \dots, T_{k-1}] \leq \frac{2}{e^{\alpha c_k} + e^{-\alpha c_k}} \leq \exp(c_k^2 \alpha^2 / 2).$$

(If $Y_k(T_1, \dots, T_m) = y$ for some y with $|y| > c_k$, we take $0 > a > 1$

such that

$$y = (1 - a)(-c_k) + ac_k$$

and define

$$Y_{*}^{(T_1, \dots, T_m)} = \left\{ \begin{array}{l} Y \text{ if } Y_k \neq y \\ c_k w / \text{pr} = a, -c_k w / \text{pr} = 1 - a. \end{array} \right.$$

Then

$$E[e^{\alpha Y_k} | T_1, \dots, T_{k-1}] \leq E[e^{\alpha Y_{*}^{(T_1, \dots, T_{k-1})}}].$$

For $X = X(T_1, \dots, T_m)$ and any possible value a of T_i ,

$$X(i; a) := X(T_1, \dots, T_{i-1}, a, T_{i+1}, \dots, T_m).$$

Corollary If

$$|X(i; a) - X(i; b)| \leq c_i \forall i, a, b,$$

(for all $T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_m$), then

$$\Pr \left[|X - E[X]| \geq \lambda \right] \leq 2 \exp \left(- \frac{2 \sum_{k=1}^n c_k^2}{\lambda^2} \right)$$

for all $\lambda > 0$.

Effect of i :

$$c_i = \max_{a, b} \|X(i; a) - X(i; b)\|_\infty.$$

More generally, suppose

T_1, \dots, T_m : independent RV's

with

$$\Pr[T_i = 0] = 1 - p_i \quad \text{and} \quad \Pr[T_i = 1] = p_i.$$

If $X = X(T_1, \dots, T_m)$ satisfies

$$|X(i;a) - X(i;b)| \leq c_i, \quad \forall i, a, b,$$

(for all $T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_m$), then for $\sigma^2 = \sum_{m^{i=1}} p_i (1 - p_i) c_i^2$ and $\lambda > 0$ with $2\lambda \max_i c_i > \sigma^2$,

$$\Pr[|X - E[X]| \geq \lambda] \leq 2e^{-\lambda^2/(4\sigma^2)}.$$

Proof. Ex. (Use, if $\alpha > 0$ and $\Pr[X = c] = 1 - \Pr[X = 0] = p$, then $E[e^{\alpha(X - E[X])}] \leq 1 + \frac{\alpha^2 p(1-p)c^2}{2} + \frac{\alpha^3 p(1-p)c^3 e^{\alpha c}}{6} \leq e^{\frac{\alpha^2 p(1-p)c^2}{2} + \frac{\alpha^3 p(1-p)c^3 e^{\alpha c}}{6}}$. And set $\alpha = \lambda/\sigma^2$.)

$X := \chi(G_{n,1/2})$: the chromatic number of $G(n, 1/2)$.

Theorem (Shamir and Spencer('87))

$$\Pr\{|X - E[X]| \geq \lambda\sqrt{n} - 1\} \leq 2e^{-\lambda^2/2}.$$

Proof. Let (set-valued) RV σ_i be the set of all neighbors in $G(n, 1/2)$ with $j > i$. Clearly, $\{\sigma_i\}$ are independent and

$$|X(i; a) - X(i; b)| \leq |X(i; j) : j > i| - X(i; \emptyset)| \leq 1.$$

Thus

$$\Pr\{|X - E[X]| \geq \lambda\sqrt{n} - 1\} \leq 2e^{-\lambda^2(n-1)/2(n-1)}.$$

Bollobás ('88) For $X = \chi(G(n, 1/2))$,

$$E[X] = \frac{2 \log_2 n}{(1 + o(1))n}.$$

Proof. Recall

$\alpha(G)$: the size of a largest independent set.

Easy (ex.)

$$\chi(G) \geq \frac{\alpha(G)}{n}.$$

For

$$E[X] \geq \frac{2 \log_2 n}{(1 + o(1))n},$$

it's enough to show that

$$\Pr[\alpha(G(n, 1/2)) \geq 2(1 + o(1)) \log_2 n] \leq n^{-2}$$

(WHY?), or equivalently, for all $\varepsilon > 0$ and $\alpha = 2(1 + \varepsilon) \log_2 n$,

$$\Pr[\exists \text{ independent set } A \text{ of size } \alpha] \leq n^{-2}.$$

(First Moment Method) For (fixed) set A of size α , as

$$\Pr[A \text{ independent}] = 2^{-\binom{\alpha}{2}}$$

we have

$$\Pr[\exists \text{ independent set } A \text{ of size } \alpha] \leq \binom{n}{\alpha} 2^{-\binom{\alpha}{2}}$$

$$\leq 2^{\alpha \log_2 n - \alpha/2 + \alpha}$$

$$= 2^{-\alpha(\varepsilon \log_2 n - 1)}$$

$$\leq n^{-2}.$$

Recall

$$\alpha = 2(1 + \varepsilon) \log_2 n.$$

For

$$E[X] \leq \frac{(1 + o(1))n}{2 \log_2 n},$$

we need a coloring.

Plan: For $\beta := 2(1 - \varepsilon) \log_2 n$

1. Take an independent set A_1 of size β in G (exists?), and assign color 1 to all vertices in it.
2. Take an independent set A_2 of β in $G \setminus A_1$, (exists?), and assign color 2 to all vertices in it.

.....

Keep doing this until $n/(\log_2 n)^2$ vertices remain uncolored, and then use extra $n/(\log_2 n)^2$ colors to color those vertices.

How many colors are used??

$$\frac{n - n / (\log_2 n)^2 (1 - \varepsilon) \log_2 n}{n} + \frac{2(1 - \varepsilon) \log_2 n}{n} \leq \frac{(1 + 2\varepsilon)n}{2 \log_2 n}$$

$$E[\# \text{ of independent set of size } \beta] = \binom{n}{\beta} 2^{-\binom{\beta}{2}}$$

$$\geq 2^{\beta(1 - \varepsilon/2) \log_2 n - \beta^2/2}$$

$$= 2^{\varepsilon \beta \log_2 n / 2}$$

(Second moment method ??)

Suppose the second moment method works,

then for step 2 ??

E.g. Two coins: X_1, X_2

Look at the two coins and take one HEAD.
If this were possible, then what is
the probability the other coin is "HEAD"?

Claim:

Every subset of size $n/(\log_2 n)^2$ contains an indep. set of size β .

Note that there are

$$\binom{n}{n/(\log_2 n)^2} \leq n^{n/(\log_2 n)^2} = 2^{n/\log_2 n}$$

such subsets.

For fixed subset W of size $n/(\log_2 n)^2$,

$X = \#$ of β -indep. subsets of W .

WANT: $\Pr[X = 0] \ll 2^{-n/\log_2 n}$

A collection $\{A_i\}$ of sets is nearly disjoint if

$$|A_i \cap A_j| \leq 1 \quad \forall i \neq j.$$

Y = the size of a largest collection of nearly disjoint β -indep. subsets of W .

WANT: $\Pr[X = 0] = \Pr[Y = 0] \gg 2^{-n/\log^2 n}$.

We will show

$$E[Y] \geq \frac{cn^2}{(\log n)^{12}}.$$

If true, then since

$$0 \leq Y(e; 0) - Y(e; 1) \leq 1,$$

we have

$$\begin{aligned} \Pr[Y = 0] &\leq \Pr[|Y - E[Y]| \geq E[Y]] \\ &\leq \exp\left(-\sum^e \frac{1}{2}\right) \\ &\leq \exp\left(-\frac{e}{2}\right) \\ &\leq \exp\left(-c^* \frac{n}{2^4 \log n}\right) \end{aligned}$$

Sketch of Proof. of

$$E[Y] \geq \frac{cn^2}{(\log n)^{12}}.$$

Let

\mathcal{A} = the set of all β -indep. subsets (of W),

\mathcal{B} = a set of (unordered) pairs $\{A, B\}$ of β -indep.

subsets with $2 \leq |A \cap B| \leq \beta - 1$.

Then

$$E[\|\mathcal{B}\|] = \Theta \left(E[\|\mathcal{A}\|]^2 (\beta \log^2 n)^4 / n^2 \right) = \Theta \left(E[\|\mathcal{A}\|]^{12} (\log n)^{12} / n^2 \right).$$

(Ex.)

For $q := E[\|A\|/2E\|B\|]$, let C be a random collection of ele. in \mathcal{A} with

$$\Pr[A \in C] = q$$

independently of all other events. Take each pair in both of \mathcal{B} and $C \times C$, and remove one of them from C . This yields a nearly disjoint collection C^* of β -indep. sets. Furthermore,

$$Y \geq |C^*| \geq |C| - |\mathcal{B} \cap C \times C| \iff$$

$$\begin{aligned} E[Y] &\geq E[|C|] - E[|\mathcal{B} \cap C \times C|] \\ &= E[\|A\|^q] - E[\|B\|^{q^2}] \\ &= \frac{E[\|A\|_2^{4E\|B\|}]}{E[\|B\|^{12}]} = \Theta\left(n^2 / (\log n)^{12}\right). \end{aligned}$$