

- Dependency Digraph

Let $\{B^i\}$ be a set of events. For each B^i , take a set of events, say $\{B^j : j \in N_+(i)\}$, such that B^j is mutually independent of all the events $\{B^k : k \notin N_+(i)\}$, meaning $\Pr[B^i \cup \bigcup_{j \in S} B^j] = \Pr[B^i] \Pr\left[\bigcup_{j \in S} B^j\right]$ for all S with $S \cup N_+(i) = \emptyset$. Draw directed edges from B^i to all of $\{B^j : j \in N_+(i)\}$.

7 Lovász Local Lemma

Formally, let

$$B_1, \dots, B_n : \text{events}, \quad V = \{1, \dots, n\}.$$

Digraph $D = (V, E)$ is a dependency graph of $\{B_i\}_{i \in V}$
if

$\forall i \in V, B_i$ is mutually independent of $\{B_j : (i, j) \notin E\}$,

$$\Pr[B^i] = \Pr[B^i] \Pr \left[\bigcup_{j \in S} B^j \right],$$

that is,

for all S with $S \cup N_+(i) = \emptyset$, where

$$\cdot \{E \ni (i, j) : j \in S\} = N_+(i)$$

Lovász Local Lemma

Suppose

for $0 < y_i < 1$. Then

$$\Pr[B_i] > y_i \quad \forall i = 1, \dots, n,$$

$$\prod_{j \in E(i)} (1 - y_j),$$

$$0 < (y_i - 1) \prod_u^{i=1} (1 - y_u) \leq \Pr[\bigcup_{u=1}^{i=1} B_u]$$

$\Pr[B_i] > D_{-5}$, and $\nabla = O(D_3)$

E.g. If $D \rightarrow \infty$, and

$$d(\varepsilon + 1) = \gamma_i \quad \text{and} \quad \frac{\nabla + 1}{1} = \gamma_i$$

Proof. Take, respectively,

$$\Pr[\bigcup_{i=1}^u B_i] < e^{-(1+o(1))pu} < 0.$$

Moreover, if $p\nabla = o(1)$, then

$$\Pr[\bigcup_{i=1}^u B_i] < \left(1 - \frac{\nabla + 1}{1}\right)^u < 0.$$

then

$$\Pr[B_i] > d \quad \text{and} \quad ep(\nabla + 1) \leq 1$$

Corollary Let Δ be the maximum degree in the dependent digraph. If

implies that

Proof. of LLL

$$\Pr[\bigcup_{i=1}^n \underline{B}_i] \leq e^{-(1+o(1))n/D_5}.$$

Notice that

$$\Pr\left[\bigcup_n \underline{B}_i\right] = \Pr[\underline{B}_1] \Pr\left[\bigcup_{i=2}^n \underline{B}_i \middle| \underline{B}_1\right] = \dots = \Pr\left[\bigcup_{i=1}^n \underline{B}_i \middle| \underline{B}_1\right]$$

NEED to prove

More generally, we claim

$$\Pr\left[\bigcup_{i=1}^n \underline{B}_i \middle| \underline{B}_1\right] \geq y_i.$$

$$\Pr\left[\bigcup_{i=1}^n \underline{B}_i \middle| \underline{B}_1\right] > y_i.$$

$$\frac{\Pr_{\substack{j \in S^2 \\ j:(i,j) \in E}} \left[\bigcup_{j \in S^1} \bar{B}_j \right]}{(y_i - 1) \prod_{j \in S^2} y_j} >$$

$$\frac{\Pr_{\substack{j \in S^2 \\ j \in S^1}} \left[\bigcup_{j \in S^1} \bar{B}_j \right]}{\Pr[B_i]} > \Pr[B_i]$$

we have

$$\Pr[B_i \cup \bigcup_{j \in S^2} \bar{B}_j] \leq \Pr[B_i \cup \bigcup_{j \in S^2} \bar{B}_j] = \Pr[B_i] \Pr \left[\bigcup_{j \in S^2} \bar{B}_j \right],$$

Let $S^1 = S \cup N_+$ and $S^2 = S \setminus S^1$, since

$$\Pr[B_i \cup \bigcup_{j \in S^2} \bar{B}_j] = \frac{\Pr \left[\bigcup_{j \in S^2} \bar{B}_j \right]}{\Pr[B_i \cup \bigcup_{j \in S^2} \bar{B}_j]}$$

for all S , by induction on $|S|$. If $S = \emptyset$, OK. For $|S| \geq 1$,

□

$$\begin{aligned}
 & \cdot (\ell_h - 1) \prod_{j \in S^1} \lesssim (\ell_h - 1) \prod_{j \in S^1} \lesssim \\
 & \Pr \left[\bigcup_{j=1}^{\ell_h} B_j \cup \bigcup_{j=r+1}^{\ell_h} \left| \bigcup_{j=r+1}^{\ell_h} B_j \right. \right] \\
 & \dots \dots \\
 & \times \left[\Pr \left[\bigcup_{j=1}^{\ell_h} B_j \cup \bigcup_{j=r+1}^{\ell_h} B_j \right] \right] \\
 & \times \left[\Pr \left[\bigcup_{j=1}^{\ell_h} B_j \right] \right] = \left[\Pr \left[\bigcup_{j \in S^1} B_j \right] \right]
 \end{aligned}$$

The induction hypothesis yields, for $S^1 = \{j_1, \dots, j_r\}$,

Theorem For $k \geq 9$,

any k -regular k -uniform hypergraph has property B.

Proof. Randomly two color all vertices with

$$\Pr[v \text{ is RED}] = \Pr[v \text{ is Blue}] = 1/2.$$

Then for the event

B_e : “ e is monochromatic”

$$\Pr[B_e] = 2^{1-k} =: p.$$

Set $B_e \leftarrow B_f$ if $e \cup f \neq \emptyset$. Since $\Delta \leq k^2$,

$$\Pr[B_e] \leq 2^{1-k}(\Delta + 1) \leq 1.$$

$\Pr[\cup_e B_e] < 0$, that is, \exists a proper 2-coloring.

Hence, LLL gives

$$0.7 \left(\frac{\ln k}{k^{1/2}} \right) 2^k \leq f(k) \leq ck 2^k.$$

edges } , we had

Recall, for $f(k) = \min\{m : \exists \text{ non-2-colorable } k\text{-uniform } H \text{ with } m$

Use random 2-coloring and apply LLL.

Proof. (Ex.) For each edge e , delete a largest degree vertex of e .

has Property B .

every simple k -uniform hypergraph H with $|H| \leq C4^k/k^3$

Theorem (Erdős & Lovasz '75) There exists $c < 0$ so that

A hypergraph $H = (V, E)$ is simple if $\text{codeg}(u, u) \leq 1$.

the number of edges containing both of u and w .

the **codegree** $\text{codeg}(u, w)$ of distinct vertices u, w is

For a hypergraph $H = (V, E)$,

- van der Waerden number $W(k)$
- Arithmetic Progression (AP) with k terms in $\{1, \dots, n\}$
- $a, a + p, a + 2p, \dots, a + (k - 1)p \in \{1, \dots, n\}$

Let $W(k)$ be the least n so that, if $\{1, \dots, n\}$ is two-colored,

\exists a monochromatic AP with k terms

$$W(k) \geq 2^{k/2}.$$

The first moment method gave

$$W(k) \text{ is FINITE for any } k.$$

van der Waerden (27)

$$W(3) = 9, W(4) = 35, W(5) = 178, \dots$$

□

and $\Delta \leq k^2 n/(k-1)$.

$$\Pr[B^S = 2^{1-k}]$$

Then

B^S : “ S is monoch.”

k -term AP S in $\{1, \dots, n\}$, let

monoch. k -term AP. Consider the random 2-coloring and, for each

Proof. We need to find a 2-coloring of $\{1, \dots, n\}$ that has no

$$W(k) \gtrsim 2^{k-1}/ek.$$

That is,

$$W(k) < u.$$

then

$$e(nk^2/(k-1) + 1)2^{1-k} > 1,$$

Theorem If

$$\Pr[|X - E[X]| > 2e^{-\chi^2_2/(4\sigma_2^2)}].$$

with $2 \max_i c_i < \sigma_2^2$,
 $T_1, \dots, T_{i-1}, T_i, \dots, T_m$, then for $\sigma_2 = \sqrt{d_i(1-d_i)c_i}$ and $\chi < 0$
 (recall $X(i; a) := X(T_1, \dots, T_{i-1}, a, T_i, \dots, T_m)$), for all

$$q, a, i, c_i, \Delta_i, |(q(i; a)X - (a(i)X)|$$

If $X(T_1, \dots, T_m)$ satisfies

$$\Pr[T_i = 1] = 1 - d_i \text{ and } \Pr[T_i = 1] = 0 =$$

Let T_1, \dots, T_m be independent RV's with

- Generalized martingale inequality

Let Δ be the maximum degree in the dependent digraph. If

Corollary of LLT

$$\Pr[B_i] \geq d \text{ with } e^d(\Delta + 1) \leq 1$$

$$\cdot <_u \left(\frac{1 + \Delta}{1} - 1 \right) \geq \Pr[\cup_{i=1}^n B_i]$$

then

$$\Pr[\cup_{i=1}^n B_i] <_e e^{-(1+o(1))}$$

Moreover, if $d\Delta = o(1)$, then

8 Incremental Random Method (Nibble)

For a hypergraph $H = (V, E)$, the **codegree** $\text{codeg}(u, w)$ of distinct vertices u, w is the number of edges containing both of u and w .

A hypergraph $H = (V, E)$ is **simple** if $\text{codeg}(u, w) \leq 1$.

Cover C of H is a set of edges whose union is V .

EASY: For a k -uniform hypergraph, as each edge in C covers k

vertices, $|C| \geq n/k$.

: nearly optimal cover if $|C| = (1 + o(1))n/k$

C : optimal cover if $|C| = n/k$

Is there a nearly PM??

\exists nearly optimal cover $\Leftrightarrow \exists$ nearly PM

and (for fixed k and $n \leftarrow \infty$)

optimal cover = PM

EASY (Ex.):

M : perfect matching (PM) if $|M| = u/k$
 $: \text{nearly PM if } |M| = (1 - o(1))u/k$

$|M| > u/k.$

Since each edge contains k vertices,

Matching M of H : a set of pairwise disjoint edges.

method

Names: Semi-random method, Nibble Method, Dynamic random

- **History of Incremental Random Method**

First used by AKS to prove

Theorem (Ajtai, Komlós and Szemerédi [81]) If G has no

triangle, then

$$\alpha(G) \geq c \frac{n \log t}{t}.$$

This implies that

$$R(3, t) > \frac{\log t}{c t^2}.$$

“Rodl Nibble”

(This was a conjecture of Erdős and Hanani.)

where the $o(1)$ -term tends to 0 as n tends to infinity.

$$\cdot \frac{\binom{t}{k}}{\binom{t}{n}} ((1 - o(1)) \leq |S(t, k, n)| S(t, k, n)$$

$S(t, k, n)$ with

Theorem (Rodl, '85) For fixed t and k , \in partial Steiner system

Known to many people by Rodl, who proved

$$\cdot \frac{k}{|V|}((1-o(1)) \leq |M|$$

then \in matching M with

$$\max_{x,y} \text{codeg}(x,y) = o(D)$$

If

hypergraph.

Theorem (Pippenger '87) Let H be k -uniform D -regular

and

Frankl & Rodl ('85)

Generalized to hypergraph by

Applied to (sparse) graph colorings:

Kahn & Kayll ('94) (fractional coloring vs. list-coloring)

Kahn ('92) (for list-coloring)

Pippenger & Spencer ('89)

Extended to edge colorings

Alon, Bollobás, K & Vu

Generalized (for non-constant k) by

K (for Steinier System)

Vu

Kostochka & Rodl

Alon, K, & Spencer ('96)

Grable ('94)

The $o(1)$ term has been improved by

Note that for any graph

$$\chi(G) \leq \Delta + 1.$$

Brooks: for connected G

$$\nabla \geq \chi(G)$$

if $G \neq K_n$, odd cycle.

If G has no triangle ??

Vizing (89)

$$\cdot \frac{(\mathcal{G}) \nabla \log \text{lo} \mathcal{G}}{(\mathcal{G}) \nabla \mathcal{G}} \geq (\mathcal{G})^i \chi$$

Theorem (Johansson '94) If \mathcal{G} has no triangle, then

$$\frac{(\mathcal{G}) \nabla \log \text{lo} \mathcal{G}}{(\mathcal{G}) \nabla \mathcal{G}} ((1 + o(1)) \leq (\mathcal{G})^i \chi$$

If \mathcal{G} has no cycle of length ≤ 4 , then

Theorem (K '93)

$K(95)$: Ramsey number $R(3, t)$

Applied also to

$$R(3, t) \geq \frac{\log t}{ct^2}.$$

Recall: AKS \iff

$$R(3, t) \leq \frac{\log t}{ct^2}.$$

Mollo & Reed ('98): coloring and total coloring

and

K & Vu: complete arc of a projective plane

and

Recall

Then \exists nearly optimal cover and nearly PM.

Theorem For simple D -reg. k -unif. hypergraph

Theorem For fixed k and $D \rightarrow \infty$, if

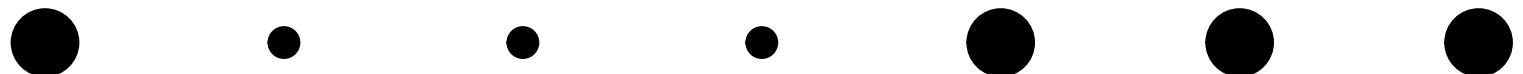
Randomly order all vertices of G ,

(A probabilistic) Proof.

$$\frac{1 + t(G)}{u} \geq \alpha(G)$$

$$\text{average degree } t(G) = \frac{1}{n} \sum_{v \in V} d(v)$$

Theorem (Turán) For a graph $G = (V, E)$ with $|V| = n$ and the



$$\left. \begin{array}{ll} 0 & \text{if } v \notin I \\ 1 & \text{if } v \in I \end{array} \right\} = (I \ni v)$$

where

$$\cdot [I \ni v] \sum^v = [(I \ni v) \mathbb{1}] E = [|I|] E$$

Note that

$$\frac{t+1}{u} \leq [|I|] E$$

Let I be the independent set obtained. Enough to show that

Take the next undeleted vertex and do the same.

Take the first vertex v_1 and delete all verts in its nbr $N(v)$.

If a vertex v precedes all vertices in $N(v)$ then

$$v \in I.$$

The corresponding probability is

$$\frac{1 + (\alpha)p}{1}$$

$$\Pr[v \in I] \geq \frac{1 + (\alpha)p}{1}$$

We have

$$\frac{1 + (\alpha)p}{u} = \frac{1 + (\alpha)p}{\sum_{i=1}^n (1/u)} \leq \frac{1 + (\alpha)p}{1} \sum_{i=1}^n \leq E[I]$$

Thus

□

by Jensen's Ineq..

$$\cdot > \varepsilon \leq 0.01/k^2.$$

Proof. Let

Then \exists nearly optimal cover and nearly PM.

$H = (V, H)$: simple D -reg. k -unif. hypergraph

Theorem For fixed k and $D \rightarrow \infty$, if

we have

$$D_* = \underline{\lambda} D^{\log D} + \overline{\lambda} D^{\log D}/2,$$

and for

$$V_* = V \setminus \bigcup_{e \in C_1} e, \quad \text{and} \quad H_* = \{e \in H : e \cap V_* = \emptyset\},$$

For the subhypergraph $H_* = (V_*, H_*)$ consists of the set V_* of all uncoveted vertices and the set of H_* of surviving edges, i.e., with the following properties.

• **PLAN:** (randomly) construct a set C_1 , called a partial cover, of $(1 + o(\varepsilon))\varepsilon n/k$ edges which covers (approx.) $(1 - e^{-\varepsilon})n$ vertices

Note that if C_1 were an optimal cover of size $\epsilon n/k$, then it would

cover *en vertices*. Since

$$(1 - \epsilon)O(\epsilon^2) = u((\epsilon^2)O + \epsilon)$$

the partial cover C_1 may be called

a “nearly optimal partial cover”.

Also, for $m = \#$ of edges in H ,

$$km = (\alpha)p \sum_{i=1}^n = \alpha D n$$

$$m = nD/k.$$

implies

$$E[\cup_{e \in C_1} e] = \sum_u \Pr[u \text{ is covered}] = (1 - O(\frac{D}{k}))O + u(1 - e^{-\frac{D}{k}})$$

Thus the expected # of covered vertices is

$$\Pr[v \in V_*] = (1 - e^{-\frac{D}{k}})^m = (1 - O(\frac{1}{k}))^m$$

and the probability $v \in V$ is not covered (by C_1) is

$$\frac{k}{n} = \frac{k}{D} \cdot \frac{D}{m} = e^{-\frac{D}{k}}$$

Then

- **Probability & Expectation**

and let C_1 denote the set of selected edges.

with probability $d = \varepsilon/D$

Independently select each edge

- **Random construction**

Finally, conditioned on $v \in V_*$, the probability that an edge e containing v survives is

$$\Pr[e \in H \mid v] = [_*A \ni a | (a)_*p]E$$

and hence

$$\cdot D_{\varepsilon(1-k)-\vartheta}((\frac{d}{1})O + 1) = [_*A \ni a | (a)_*p]E$$

• Concentration
Let $X = |C_1|$. Then

$$|X - (0; 1)| \leq 1,$$

$$\sum_{i=2}^e p_i = d - 1.$$

For $\alpha = u/\sqrt{D} > \frac{pm}{\epsilon n} = \frac{2k}{\epsilon n}$, we have

$$\Pr_D \left[\frac{\sqrt{D}}{u} \geq [X]E - X \right] \geq 1 - e^{-\alpha^2}.$$

and

Let $Y = \#$ of covered vertices = $|\cup_{e \in C_1} e|$. Then

$$k \geq |(e; 1) - (0; e)| Y$$

For $2\alpha = 2n/\sqrt{D} > p m k^2 = \varepsilon k n$, we have

$$\varrho^2 > p m k^2.$$

and

$$\Pr[C_1] \geq \frac{\sqrt{D}}{u} + \frac{k}{u\varepsilon} \text{ and } \Pr[\cup_{e \in C_1} e \in C_1] \geq 1 - 2e^{-\alpha u/D}.$$

Thus

$$\Pr \left[|d_* - E[d_*]| \geq \sqrt{D \log D / 2} \right] \leq \exp \left(- \frac{4e k^3 D}{\sqrt{D \log D / 2}} \right) < e^{-6 \log D} = D^{-6}.$$

For $2\lambda = \sqrt{D \log D} < \epsilon k^2 D$, we have

$$\varphi_2 \leq p k_2 k D_2 = \epsilon k^3 D.$$

Since there are $\leq k D_2$ edges satisfying the first condition,

$$\emptyset \neq e \cup f \in \mathcal{F} \quad \left. \begin{array}{ll} n \in f \cap e, & k \\ \text{if } & \end{array} \right\} \geq |(q; \epsilon)_*^n p - (a; \epsilon)_*^n p|$$

Finally, let $d_* = d_*(v)$. Then conditioned on $v \in V_*$

Recall

$$\epsilon \in C_1$$

$$\Pr[|C_1| \leq \frac{k}{\epsilon u} + \frac{\sqrt{D}}{u}] \text{ and}$$

$$|e| \leq (1 - e^{-\epsilon})^n - 2e^{-n/D}.$$

$$\cdot^{D/n} = e^{-(1+o(1))n/D}$$

$$\Pr[d_* - E[d_*] > \sqrt{D \log D / 2}, A_u \in V_*] \geq |[{}^a p]_E - [{}^a p]|$$

Therefore,

$$\Delta \geq kD^{\frac{1}{3}}, \text{ which yields } b \Delta \gg 1.$$

The maximum degree Δ of the dependent digraph,

- Homework 1: Exercises in pages 31, 40, 43, 52, 88, 108, 110, 112, 126 (Due 2/2/07)
- Z. Füredi, Random Polytopes in the d -Dimensional Cube, Discrete Comput. Geom. 1: 315-319 (1986).
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- Radhakrishnan & Srinivasan, Improved Bounds and Algorithms for Hypergraph 2-coloring, Random Structures & Algorithms 16, 4-32, (2000).
- N. Alon, J. Kim and J. Spencer Nearly Perfect Matchings in Regular Simple Hypergraphs. Israel J. of Math. 100 (1997), 171-187.

List of Papers