8.2 Asymptotics of Ramsey number $R(3, t)$

$$R(s,t) \quad := \quad \min\{n : \text{for every } G \text{ on } n \text{ vertices}, \\ \omega(G) \geq s \text{ or } \alpha(G) \geq t\}$$

EASY

$$R(s,t) = R(t,s), \qquad R(2,t) = t$$

Greenwood & Gleason ('55):

$$R(3,3) = 6, \quad R(3,4) = 9,$$

$$R(3,5) = 14, \ R(4,4) = 18$$

MORE:

$$R(3,6) = 18, \ R(3,7) = 23, \ R(3,8) = 28, \\ R(3,9) = 36, \ R(4,5) = 25,$$

• Ramsey number $R(3,t)$

Erdős('61)

$$R(3,t) \geq c_1 \frac{t^2}{(\log t)^2} \ ,$$

Graver & Yackel('68)

$$R(3,t) \leq c_2 \frac{t^2 \log\log t}{\log t} \ .$$

Ajtai, Komlós and Szemerédi ('81)

$$R(3,t) \leq c_3 \frac{t^2}{\log t}$$

(removed the "$\log\log t$" factor).

Little improvement on lower bound

Spencer ('77), Bollobás ('85),

Erdős, Suen and Winkler ('93)

Krivelevich ('94)

simplified its proof and/or increased constant

Theorem (K '95)

$$R(3,t) \geq c(1-o(1))\frac{t^2}{\log t},$$

with $c = 1/162 = 1/(2 \cdot 9^2)$.

Idea of the proof of

$$R(3,t) \geq c(1-o(1))\frac{t^2}{\log t}.$$

Recall

$$R(3,t) := \min\{n : \text{for every } G_n,\ \ \omega(G_n) \geq 3 \text{ or } \alpha(G_n) \geq t\}$$

Enough to show

$$\exists \text{ triangle-free } G_n \text{ for which}$$

$$\alpha(G_n) \leq 9\sqrt{n \log n}$$

for sufficiently large n.

• **Random Greedy Methods vs. Nibble Methods**

· Random Greedy (or One-by-One) Construction

1. Randomly order all edges in K_n
($\exists$ $\binom{n}{2}!$ possible ways)

2. Choose edges greedily according to the random order. (An edge cannot be chosen only if it makes a triangle with previously chosen edges.)

• Incremental Random Method (Nibble method)

Let

$$\Gamma_0 : \text{ the set of all } \binom{n}{2} \text{ edges.}$$

Define

$$\text{a random subset } X_1 \text{ of } \Gamma_0\text{:}$$

$$\Pr[e \in X_1] := \varepsilon/\sqrt{n} \quad \text{for all } e \in \Gamma_0$$

independently.

Take any "maximal" (under $\subseteq$) family $\mathcal{F}_1$ of edge disjoint triangles in X_1.

Deleting all edges belong to triangles in $\mathcal{F}$ we obtain a Δ-free graph G_1 on n vertices.

$$\text{An edge } e \in \Gamma_0 \text{ survives}$$
$$\text{if } e \notin X_1 \text{ and}$$
$$\text{there no edges } f, g \in Y_1 := X_1 \text{ s.t. } efg \text{ is a triangle.}$$

Let

$$\Gamma_1 \text{ be the set of all surviving edges.}$$

Define

$$\text{a random subset } X_2 \text{ of } \Gamma_1\text{:}$$

$$\Pr[e \in X_2] := \varepsilon/\sqrt{n} \quad \text{for all } e \in \Gamma_1$$

independently.

Forbidden pairs of edges:

$$\Lambda_2 := \{e_{uv}e_{vw} : e_{uv}, e_{vw} \subseteq X_2, e_{wu} \in Y_1\} \ ,$$

where $e_{vw} := \{v, w\}$.

Forbidden triples of edges:

$$\Delta_2 := \{e_{uv}e_{vw}e_{wu} : e_{uv}, e_{vw}, e_{wu} \subseteq X_2\}$$

Take any "maximal" (under $\subseteq$) family $\mathcal{F}_2$ of edge disjoint forbidden pairs and triples in $\Lambda_2 \cup \Delta_2$.

Deleting all edges belong to pairs and triangles in $\mathcal{F}_2$
we obtain a Δ-free graph

$$G_2 = G_1 \cup \left(X_2 \setminus \cup_{F \in \mathcal{F}_2} F\right)$$

on n vertices.

An edge $e \in \Gamma_1$ survives
if $e \notin X_2$ and
there no edges $f, g \in Y_2 := Y_1 \cup X_2$ s.t.
efg is a triangle.

Let

Γ_2 be the set of all surviving edges.

At step i, define

a random subset X_i of Γ_{i-1}:

$$\Pr[e \in X_i] := \varepsilon/\sqrt{n} \quad \text{for all } e \in \Gamma_{i-1}$$

independently.

Forbidden pairs of edges:

$$\Lambda_i := \{e_{uv}e_{vw} : e_{uv}, e_{vw} \subseteq X_i, e_{wu} \in Y_{i-1}\} .$$

Forbidden triples of edges:

$$\Delta_i := \{e_{uv}e_{vw}e_{wu} : e_{uv}, e_{vw}, e_{wu} \subseteq X_i\}$$

Take any "maximal" (under $\subseteq$) family $\mathcal{F}_i$ of edge disjoint forbidden pairs and triples in $\Lambda_i \cup \Delta_i$.

Deleting all edges belong to pairs and triangles in $\mathcal{F}_i$,
we obtain a Δ-free graph

$$G_i = G_{i-1} \cup (X_i \setminus \cup_{F \in \mathcal{F}_i} F).$$

An edge $e \in \Gamma_i$ survives
if $e \notin X_i$ and
there no edges $f, g \in Y_i := Y_{i-1} \cup X_i$ s.t.
efg is a triangle.

Let Γ_{i+1} be the set of all surviving edges.

FACT: as $\varepsilon \longrightarrow 0$

$$\frac{|\bigcup_{F \in \mathcal{F}_i} F|}{|X_i|} \longrightarrow 0$$

So, small enough $\varepsilon \Longrightarrow Y_i \approx \mathcal{E}(G_i)$

$$\varepsilon = (\log n)^{-2}, \;\; \# \text{ of steps} \approx n^{1/17}$$

$\mathcal{I}_i$: the collection of all independent sets in G_i of size t, i.e.,

$$\mathcal{I}_i = \{T : |T| = t,\ T \text{ indepen. in } G_i\},$$

where $t := \lceil 9\sqrt{n \log n}\, \rceil$.

$$\text{STOP when } |\mathcal{I}_i| < 1.$$

possible ???

Let $\Gamma_i(T)$ be the set of surviving edges in T. Then WANT

Prop. 7.

$$|\Gamma_i(T)| \geq b_i \mu_i \binom{t}{2} .$$

• Probabilities

Suppose $\exists (unknown) \psi$ satisfying

$$\Pr[e \in Y_i] \approx \Pr[e \in \mathcal{E}(G_i)] = \frac{\psi(i\varepsilon)}{\sqrt{n}},$$

for all i. Then

$$b_i := \Pr[e \in \Gamma_i] =???$$

Consider a random graph $G(n, p)$ with edge prob. p, where

$$p = \frac{\psi(i\varepsilon)}{\sqrt{n}} \ .$$

Regarding G_i as $G(n,p)$, $p = \frac{\psi(i\varepsilon)}{\sqrt{n}}$,

$$\begin{aligned}
\Pr[e \in \Gamma_i] &= (1-p^2)^{n-2} \\
&= \left(1 - \left(\frac{\psi(i\varepsilon)}{\sqrt{n}}\right)^2\right)^{n-2} \\
&\approx \exp(-\psi^2(i\varepsilon))
\end{aligned}$$

On the other hand, we know

$$|\mathcal{E}(G_{i+1})| \approx |\mathcal{E}(G_i)| + |X_{i+1}|$$

and, in expectations,

$$|\mathcal{E}(G_{i+1})| \approx \frac{\psi((i+1)\varepsilon)}{\sqrt{n}}\binom{n}{2} \approx \frac{\psi((i+1)\varepsilon)n^{3/2}}{2},$$

$$|\mathcal{E}(G_i)| \approx \frac{\psi(i\varepsilon)}{\sqrt{n}}\binom{n}{2} \approx \frac{\psi(i\varepsilon)n^{3/2}}{2}$$

and

$$
\begin{aligned}
|X_{i+1}| &\approx \frac{\varepsilon}{\sqrt{n}} |\Gamma_i| \\
&\approx \frac{\varepsilon}{\sqrt{n}} \exp(-\psi^2(i\varepsilon)) \binom{n}{2} \\
&\approx \frac{\varepsilon \exp(-\psi^2(i\varepsilon)) n^{3/2}}{2} \; .
\end{aligned}
$$

As $\mathcal{E}(G_{i+1}) \approx \mathcal{E}(G_i) + |X_i|$, we have

$$
\frac{\psi((i+1)\varepsilon) n^{3/2}}{2} \approx \frac{\psi(i\varepsilon) n^{3/2}}{2} + \frac{\varepsilon \exp(-\psi^2(i\varepsilon)) n^{3/2}}{2}.
$$

In other words,

$$\psi((i+1)\varepsilon) \approx \psi(i\varepsilon) + \varepsilon \exp(-\psi^2(i\varepsilon))$$

or

$$\psi'(x) \approx \exp(-\psi^2(x)) \ .$$

Define the function $\psi(x)$:

$$x = \int_0^{\psi(x)} e^{\xi^2} \ d\xi \ .$$

Notice that

$$\psi(x) \sim \sqrt{\log x} \ .$$

Let

$$\begin{aligned} b_i &:= \psi'(i\varepsilon) = \exp(-\psi^2(i\varepsilon)), \\ a_i &:= \sum_{j=0}^{i-1} b_j \varepsilon = \sum_{j=0}^{i-1} \psi'(j\varepsilon)\varepsilon = a_{_{i-1}} + b_{_{i-1}}\varepsilon \\ &\approx \psi(i\varepsilon) \end{aligned}$$

where $\varepsilon := (\log n)^{-2}$, and

$$\mu_i := 1 - 20a_i\varepsilon - \frac{a_i}{3\sqrt{\log n}}$$

For $A, B \subseteq V$, let

$$\Gamma_i(A, B) := \{e_{vw} \in \Gamma_i : v \in A, w \in B\} \;\; \text{and} \;\; \Gamma_i(A) := \Gamma_i(A, A) \; .$$

Prop. 1. $d_{\mathrm{Y}_i}(v) \le a_i\sqrt{n} + in^{1/4}(\log n)^2$.

Prop. 2. $d_{\Gamma_i}(v) \le b_i n$.

Prop. 3. $|N_{\mathrm{Y}_i}(v) \cap N_{\mathrm{Y}_i}(w)| \le 3i\log n$.

Prop. 4. $d_{\Lambda_i}(e_{vw}, v) \le b_i(a_i + 5\varepsilon)\sqrt{n}$.

Prop. 5. $d_{\Delta_i}(e) \le b_i^2 n$.

Prop. 6. For $A \cap B = \emptyset$ with $|A|, |B| \ge \varepsilon^2 b_i^2\sqrt{n}$,

$$|\Gamma_i(A, B)| \le b_i|A||B| \ .$$

$$|\Gamma_i(A)| \le b_i\binom{|A|}{2} \ .$$

Property 7. For all $T \subseteq V$ with $|T| = 9\sqrt{n \log n}$,

$$|\Gamma_i(T)| \geq b_i \mu_i \binom{t}{2} \ .$$

Let $\mathcal{I}_i$ be the set of independent sets of size $9\sqrt{n \log n}$ in G_i.

Property 8.

$$|\mathcal{I}_i| \leq n^i \binom{n}{t} \exp\Bigg(-(1-\varepsilon) \sum_{j=0}^{i-1} \frac{b_j \mu_j \varepsilon}{\sqrt{n}} \binom{t}{2} \Bigg) \ ,$$

Definitions

For given (Y_i, Γ_i, G_i), set

$$\begin{aligned} \Lambda_i &:= \{ef \subseteq \Gamma_i : \exists\ g \in Y_i \text{s.t.} efg \in \Delta_0\} \\ \Delta_i &:= \{efg \subseteq \Gamma_i : efg \in \Delta_0\}\ , \end{aligned}$$

and

$$N_{Y_i}(v) := \{w \in V : e_{vw} \in Y_i\}\ ,\ d_{Y_i}(v) := |N_{Y_i}(v)|\ .$$

Given $v \in V$, let

$$\begin{aligned} N_{\Gamma_i}(v) &:= \{w \in V : e_{vw} \in \Gamma_i\} \\ \mathcal{M}_{\Gamma_i}(v) &:= \{e_{vw} : e_{vw} \in \Gamma_i\} \\ d_{\Gamma_i}(v) &:= |N_{\Gamma_i}(v)| = |\mathcal{M}_{\Gamma_i}(v)|\ . \end{aligned}$$

Also, for $e_{vw} \in \Gamma_i$,

$$\begin{aligned} N_{\Lambda_i}(e_{vw}, v) &:= \{u \in V : e_{uv} e_{vw} \in \Lambda_i\} \\ \mathcal{M}_{\Lambda_i}(e_{vw}, v) &:= \{e_{uv} \in \Gamma_i : e_{uv} e_{vw} \in \Lambda_i\} \\ d_{\Lambda_i}(e_{vw}, v) &:= |N_{\Lambda_i}(e_{vw}, v)| = |\mathcal{M}_{\Lambda_i}(e_{vw}, v)| \ , \end{aligned}$$

and

$$\begin{aligned} N_{\Lambda_i}(e_{vw}) &:= N_{\Lambda_i}(e_{vw}, v) \cup N_{\Lambda_i}(e_{vw}, w) \\ \mathcal{M}_{\Lambda_i}(e_{vw}) &:= \mathcal{M}_{\Lambda_i}(e_{vw}, v) \cup \mathcal{M}_{\Lambda_i}(e_{vw}, w) \ . \end{aligned}$$

Finally,

$$N_{\Delta_i}(e_{vw}) := \{u \in V : e_{uv} e_{vw} e_{wu} \subseteq \Gamma_i, \} \ .$$

All properties would seem quite natural to expect.

For example, we would expect

$$
\begin{aligned}
& d_{\Lambda_i}(e_{vw}, v) \\
:= \quad & \sum_{u \in V \setminus e_{vw}} 1(e_{wu} \in Y_i \ \text{ and } \ e_{uv} \in \Gamma_i) \\
\approx \quad & n \Pr[e_{wu} \in Y_i \ \text{ and } \ e_{uv} \in \Gamma_i] \\
\approx \quad & n \Pr[e_{wu} \in Y_i] \Pr[e_{wu} \in \Gamma_i] \\
\approx \quad & n(a_i/\sqrt{n}) b_i = a_i b_i \sqrt{n}
\end{aligned}
$$

HOW TO PROVE

Prop. 1. $d_{Y_i}(v) \leq a_i\sqrt{n} + in^{1/4}(\log n)^2$.

Prop. 2. $d_{\Gamma_i}(v) \leq b_i n$.

Prop. 3. $|N_{Y_i}(v) \cap N_{Y_i}(w)| \leq 3i \log n$.

....

(a) Show the properties at the level of expectations.

(b) Prove that the random variables $d_{Y_i}(v)$ etc. are highly concentrated near their means. For example,

$$\Pr\left[d_{Y_i}(v)E[d_{Y_i}(v)] \geq \delta E[d_{Y_i}(v)]\right] \leq e^{-(\log n)^2} \ .$$

For (b), we need martingale inequalities.

Except **Property 7**:

For all $T \subseteq V$ with $|T| = 9\sqrt{n \log n}$,

$$|\Gamma_i(T)| \geq b_i \mu_i \binom{t}{2} \ .$$

Lemma 8.1 *The following three conditions hold simultaneously with probability at least* $1 - 3/n^2$*:*

(i) For all $v \in V$, $|N_{X_{i+1}}(v)| \leq b_i \varepsilon \sqrt{n} + n^{1/4} \log n$*;*

(ii) For all $v \neq w \in V$, $|N_{G_i}(v) \cap N_{X_{i+1}}(w)| \leq \log n$*;*

(iii) For all $v \neq w \in V$, $|N_{X_{i+1}}(v) \cap N_{X_{i+1}}(w)| \leq \log n$*.*

Remark

1. A better constant could be possible:

Setting $p_i := \dfrac{\varepsilon}{b_i\sqrt{n}}$.

BUT "More Complicated".

2. $R(4,t) =??$

· Probably too many properties

· NOT enough independence:

HARD to guess the parameters

· ONLY $(\log t)^x$ improvement:

$$t^{5/2} \lesssim R(4,t) \lesssim t^3$$

Giant Component of Random graph & 2-SATisfiability Problem

- Random Graph $G(n, p)$:

each of $\binom{n}{2}$ edges is independently
in $G(n, p)$ with probability p

$p = 1$: complete graph $p = 0$: empty graph

Expected number of edges

$$p\binom{n}{2}$$

For fixed $G = (V, E)$,

$$\Pr[G(n, p) = G] = p^{|E|}(1-p)^{\binom{n}{2}-|E|}$$

W_i : the size of the i^{th} largest component of $G(n,p)$

- Erdős & Rényi ('60, '61)

$$W_1 \quad \begin{cases} \leq c \log n, & p = (1-\epsilon)/n \\ = \Theta(n^{2/3}), & p \sim 1/n \\ \sim f(\epsilon)n, & p = (1+\varepsilon)/n \end{cases}$$

($\epsilon > 0$), where $f(\varepsilon)$ is the positive sol. of

$$1 - f(\varepsilon) = \exp(-(1+\varepsilon) f(\varepsilon))$$

If ε is small,

$$f(\varepsilon) \sim 2\varepsilon$$

What if

$$p = \frac{1 \pm n^{-\delta}}{n} ??$$

Bollobás ('84), Łuczak ('90),

Janson, Knuth, Łuczak & Pittel ('94)

For $p = \frac{1-\lambda n^{-1/3}}{n}$

$$W_1 = \Theta(n^{2/3} \log \lambda / \lambda^2)$$

$$W_2 = \Theta(n^{2/3} \log \lambda / \lambda^2),$$

particularly

$$\frac{W_1}{n^{2/3}} \to 0, \quad \text{as } \lambda \to \infty$$

For $p = \frac{1+\lambda n^{-1/3}}{n}$

$$W_1 \approx 2\lambda n^{2/3}$$

$$W_2 = \Theta(n^{2/3} \log \lambda / \lambda^2)$$

and hence

$$\frac{W_1}{n^{2/3}} \to \infty, \quad \text{as } \lambda \to \infty$$

$$\frac{W_2}{n^{2/3}} \to 0, \quad \text{as } \lambda \to \infty$$

• Random Cluster Model $H = H(n, p)$:

For fixed $G = (V, E)$,

$$\Pr[G(n,p) = G] \propto p^{|E|}(1-p)^{\binom{n}{2}-|E|} q^{c(G)}$$

where $q > 0$ and

$$c(G) = \# \text{ connected components of } G$$

(cf. $\Pr[G(n,p) = G] = p^{|E|}(1-p)^{\binom{n}{2}-|E|}$)

Potts model on K_n :

configuration $\sigma : V \to \{1, 2, ..., q\}$

$$w(\sigma) = \exp\left(\beta \sum_{\substack{i,j \in V \\ i \neq j}} \delta(\sigma(i), \sigma(j))\right)$$

where

$$\delta(\sigma(i), \sigma(j)) = \begin{cases} 1 & \text{if } \sigma(i) = \sigma(j)) \\ 0 & \text{otherwise} \end{cases}$$

Partition function

$$
\begin{aligned}
Z(\beta) &= \sum_{\sigma} w(\sigma) \\
&= \sum_{\sigma} \exp\left(\beta \sum_{\substack{i,j \in V \\ i \neq j}} \delta(\sigma(i), \sigma(j))\right)
\end{aligned}
$$

FK (Fortuin & Kasteleyn) Representation:

$$
\begin{aligned}
Z(\beta) &= \sum_{\sigma} \prod_{\substack{i,j \in V \\ i \neq j}} e^{\beta\delta(\sigma(i),\sigma(j))} \\
&= \sum_{\sigma} \prod_{\substack{i,j \in V \\ i \neq j}} \left(1 + \left(e^{\beta\delta(\sigma(i),\sigma(j))} - 1\right)\right) \\
&= \sum_{\sigma} \sum_{E} \prod_{\substack{\{i,j\} \in E \\ i \neq j}} \left(e^{\beta\delta(\sigma(i),\sigma(j))} - 1\right) \\
&= \sum_{E} \sum_{\sigma} \prod_{\substack{\{i,j\} \in E \\ i \neq j}} \left(e^{\beta\delta(\sigma(i),\sigma(j))} - 1\right)
\end{aligned}
$$

For $G = (V, E)$,

$$
\sum_{\sigma} \prod_{\substack{\{i,j\} \in E \\ i \neq j}} \left(e^{\beta\delta(\sigma(i),\sigma(j))} - 1\right) = \left(e^{\beta} - 1\right)^{|E|} q^{c(G)}
$$

$\exists \ \ \alpha_c(q)$ s.t.

$$\frac{W_1}{n} \to \begin{cases} 0 & \text{if } p \leq (\alpha_c(q) - \varepsilon)/n \\ f(\varepsilon, q) & \text{if } p = (\alpha_c(q) + \varepsilon)/n \end{cases}$$

where $f(\varepsilon, q)$ is the positive sol. of

$$\frac{1 - f(\varepsilon, q)}{1 + (q-1) f(\varepsilon, q)} = \exp(-(\alpha_c(q) + \varepsilon) f(\varepsilon, q))$$

In fact,

$$\alpha_c(q) = \begin{cases} q & 0 < q \leq 2 \\ \frac{2(q-1)\log(q-1)}{q-2} & \text{if } q > 2 \end{cases}$$

• Satisfiability

Boolean Variables: $x_1, ..., x_n \in \{0,1\}$

Negation of x: $\bar{x} = 1 - x$

$2n$ literals: $x_1, \bar{x}_1, ..., x_n, \bar{x}_n$

x and y are *strictly distinct (s.d.)*
if $x \neq y$ and $x \neq \bar{y}$

k-clause:

$$C = l_1 \bigvee \cdots \bigvee l_k$$

where $l_1, ..., l_k$ are s.d. literals

How many k-clauses??

Take k Boolean variables out of n.

Then $\exists$ two choices (negation or not) for each variable.

$$2^k \binom{n}{k}$$

k-SAT Formula:

$$F = F(x_1, ..., x_n) = C_1 \bigwedge \cdots \bigwedge C_m$$

where $C_1, ..., C_m$ are k-clauses.

F is *satisfiable* if

$$F(x_1, ..., x_n) = 1$$

for some $x_1, ..., x_n \in \{0, 1\}$

k-SAT problem: NP-Complete if $k \geq 3$

(P if $k = 2$)

- Random k-SAT $F_k(n,p)$:

Each k-clause appears in F
with probability p

Expected # of clauses

$$m = 2^k p \binom{n}{k}$$

(Goerdt '92, Chvátal & Reed '92, F. de la Vega '92) For $k = 2$,

$$\Pr[F_2 \text{ is SAT }] \to \begin{cases} 1 & \text{if } \; m/n \to c < 1 \\ 0 & \text{if } \; m/n \to c > 1 \end{cases}$$

Conjecture. For $k \geq 3$, $\exists\ \alpha(k)$ s.t.

$$\Pr[F_k \text{ is SAT }] \to \begin{cases} 1 & \text{if } \ m/n \to c < \alpha(k) \\ 0 & \text{if } \ m/n \to c > \alpha(k) \end{cases}$$

Known

$$3.14 \leq \alpha(3) \leq 4.596$$

$$c_1 2^k/k \leq \alpha(k) \leq c_2 2^k$$

Pittel:

"Y2K Problem"

Friedgut ('97) Let

$$p_n^{(-)}(\delta) = \max\{p : \Pr[F_k(n,p) \text{ is SAT }] \ \geq 1-\delta\}$$

and

$$p_n^{(+)}(\delta) = \min\{p : \Pr[F_k(n,p) \text{ is SAT }] \ \leq \delta\}$$

Then

$$(p_n^{(+)}(\delta) - p_n^{(-)}(\delta))2^k \binom{n}{k} = o(1/n)$$

For $k = 2$,

$$\Pr[F_2 \text{ is SAT }] \to \begin{cases} 1 & \text{if } \; m/n \to c < 1 \\ 0 & \text{if } \; m/n \to c > 1 \end{cases}$$

What if

$$\frac{m}{n} = 1 \pm n^{-\delta} \quad ??$$

(Bollobás, Borgs, Chayes, K, Wilson) For $m/n = 1 - \lambda n^{-1/3}$

$$\Pr[F_2 \text{ is SAT }] = 1 - \Theta(1/\lambda^3)$$

For $m/n = 1 + \lambda n^{-1/3}$

$$\Pr[F_2 \text{ is SAT }] = \exp(-\Theta(\lambda^3))$$

Spine S_F:

A satisfiable formula F *fixes* a literal x

if

x is true (i.e. $x = 1$) in all satisfying assignments.

A literal x is the *spine* S_F of a formula F

iff

$\exists$ a satisfiable subformula which fixes x.

For $m/n = 1 - \lambda n^{-1/3}$

$$\frac{E[|S_F|]}{n^{2/3}} \sim \frac{1}{2\lambda^2}$$

For $m/n = 1 + \lambda n^{-1/3}$

$$\frac{E[|S_F|]}{n^{2/3}} \sim 4\lambda$$

9 Branching Process and Giant Component

$G(n, p)$ undergoes a remarkable change at $p = 1/n$. (Erdős and Rényi, 1960)

- $p = c/n$ with $c < 1$
  - consists of small components, the largest of which is of size $\Theta(\ln n)$.
- $p = c/n$ with $c > 1$
  - forms a "giant component" of size $\Theta(n)$.

9.1 Branching Process

Imagine the following stochastic process called *branching process*.

- A unisexual universe
- Initially there is one *live* organism and no *dead* ones.
- At each time unit, we select one of the live organisms, it has Z children, and then it dies.
- Z will be Poisson with mean c.

We want to study *whether or not the process continues forever.*

More precisely,

- Let Z_i be the number of children of the organism selected at time i.
  - $Z_1, Z_2, \ldots$ be independent random variables, each with distribution Z.
- Let Y_i be the number of live organisms at time i. Then, $Y_0, Y_1, \ldots$ is given by the recursion

$$\begin{aligned} Y_0 &= 1, \\ Y_i &= Y_{i-1} + Z_i - 1, \end{aligned}$$

for $i \geq 1$.

- Let T be the least t such that $Y_t = 0$. If no such t exists, we say $T = +\infty$.
- T is the total number of organisms in the process.
- The process stops when $Y_t = 0$ but we define the recursion for all t.

Theorem *When* $E[Z] = c < 1$, *the process dies out* ($T < \infty$) *with probability* 1.

Proof.

- Since $Y_t = Z_1 + \cdots + Z_t - t + 1$,

$$\Pr[T > t] \leq \Pr[Y_t > 0] = \Pr[Z_1 + \cdots + Z_t \geq t].$$

- $Z_1 + \cdots + Z_t$ has a Poisson distribution with mean ct. Then,

$$\Pr[Z_1 + \cdots + Z_t \geq t] \leq (ce^{1-c})^t.$$

- From the fact that $ce^{1-c} < 1$ for $c < 1$,

$$\lim_{t \to \infty} \Pr[T > t] = 0,$$

  which means that $\Pr[T = \infty] = 0$.

$\square$

Theorem *When $E[Z] = c > 1$, there is a nonzero probability that the process goes on forever ($T = \infty$).*

Proof.

- As in the proof of the previous theorem,

$$\Pr[Z_1 + \cdots + Z_t \leq t] \leq (1 - \delta)^t,$$

  with $\delta > 0$.

- As $\sum_{t=1}^{\infty} (1 - \delta)^t$ converges, there is a t_0 with

$$\sum_{t=t_0}^{\infty} \Pr[Z_1 + \cdots + Z_t \leq t] < 1.$$

- Then, conditioned on $Z_1 = t_0$,

$$Y_t = t_0 + (Z_2 - 1) + \cdots + (Z_t - 1), \quad \text{for } t \geq 2,$$

and so

$$\sum_{t=2}^{\infty} \Pr[Y_t \leq 0 | Z_1 = t_0] = \sum_{t=0}^{\infty} \Pr[t_0 + Z_2 + \cdots Z_t \leq t - 1]$$

$$\leq \sum_{t=t_0+1}^{\infty} \Pr[Z_2 + \cdots Z_t \leq t - 1] < 1.$$

Therefore,

$$\Pr[T = \infty] \geq \Pr[Z_1 = t_0]\Big(1 - \sum_{t=t_0}^{\infty} \Pr[Z_1 + \cdots Z_t \leq t]\Big) > 0.$$

$\square$

Analysis using generating functions

- Let

$$p_i = \Pr[Z_1 = i] = e^{-c} c^i / i!$$

and define the generating function

$$p(x) = \sum_{i=0}^{\infty} p_i x^i = \sum_{i=0}^{\infty} e^{-c} c^i x^i / i! = e^{c(x-1)}.$$

- Let $q_i = \Pr[T = i]$ and set

$$q(x) = \sum_{i=0}^{\infty} q_i x^i.$$

- Conditioning on the first organism having s children, the generating function for the total number of offspring is

$$\begin{aligned}
\sum_{i=0}^{\infty} \Pr[T = i | Z_1 = s] x^i &= \sum_{i=0}^{\infty} \sum_{j_1 + \cdots + j_s = i-1} q_{j_1} \cdots q_{j_s} x^i \\
&= x \sum_{j=0}^{\infty} \sum_{j_1 + \cdots + j_s = j} q_{j_1} \cdots q_{j_s} x^j \\
&= x (q(x))^s.
\end{aligned}$$

- Hence

$$
\begin{aligned}
q(x) &= \sum_{i=0}^{\infty} q_i x^i \\
&= \sum_{i=0}^{\infty} \Pr[T = i] x^i \\
&= \sum_{i=0}^{\infty} \sum_{s=0}^{\infty} \Pr[Z_1 = s] \Pr[T = i | Z_1 = s] x^i \\
&= \sum_{s=0}^{\infty} \Pr[Z_1 = s] \sum_{i=0}^{\infty} \Pr[T = i | Z_1 = s] x^i \\
&= \sum_{s=0}^{\infty} p_s x q(x)^s \\
&= x \sum_{s=0}^{\infty} p_s q(x)^s = x p(q(x)).
\end{aligned}
$$

- $y_x = q(x)/x$ satisfies the functional equality $y_x = p(xy_x)$, i.e.,

$$y_x = e^{c(xy_x - 1)}.$$

- The extinction probability

$$y := \Pr[T < \infty] = \sum_{i=0}^{\infty} \Pr[T = i] = \sum_{i=0}^{\infty} q_i = q(1) = q(1)/1 = y_1$$

must satisfy

$$y = e^{c(y-1)}.$$

- For $c < 1$, $y = e^{c(y-1)}$ has the unique solution $y = 1$, corresponding to the certain extinction.
- For $c > 1$, there are two solutions, $y = 1$ and $y = y^* \in (0, 1)$.
- As $\Pr[T < \infty] < 1$, $\Pr[T < \infty] = y^*$.

- When a branching process dies, we call $H = (Z_1, \ldots, Z_T)$ the *history* of the process.
- A sequence $(z_1, \ldots, z_t)$ is a possible history if and only if the sequence y_i given by $y_0 = 1$, $y_i = y_{i-1} + z_i - 1$ has $y_i > 0$ for $0 \leq i < t$ and $y_t = 0$.
- When Z is Poisson with mean λ,

$$\Pr[H = (z_1, \ldots, z_t)] = \prod_{i=1}^{t} \frac{e^{-\lambda}\lambda^{z_i}}{z_i!} = \frac{e^{-\lambda}(\lambda e^{-\lambda})^{t-1}}{\prod_{i=1}^{t} z_i!},$$

since $z_1 + \cdots + z_t = t - 1$.

- We call $d < 1 < c$ a *conjugate pair* if

$$de^{-d} = ce^{-c}.$$

- Since $y^* = e^{c(y^*-1)}$,

$$(cy^*)e^{-cy^*} = ce^{-c},$$

so cy^* and c is a conjugate pair.

- For every history $H = (z_1, \ldots, z_t)$,

$$\begin{aligned} \Pr_c[H = (z_1, \ldots, z_t) | T < \infty] &= \frac{e^{-c}(ce^{-c})^{t-1}}{y^* \prod_{i=1}^{t} z_i!} \\ &= \frac{e^{-cy^*}(cy^* e^{-cy^*})^{t-1}}{\prod_{i=1}^{t} z_i!} \\ &= \Pr_d[H = (z_1, \ldots, z_t)], \end{aligned}$$

since $ce^{-c} = (cy^*)e^{-cy^*}$ and $y^* e^{-cy^*} = e^{-c}$.

Theorem *The branching process with mean c, conditional on extinction, has the same distribution as the branching process with mean $d = cy^*$.*

9.2 Giant Component

We define a procedure to find the component $C(v)$ containing a given vertex v in a graph $G = G(n, p)$.

- Vertices will be *live*, *dead*, or *neutral*.
- Originally v is live, all other vertices are neutral, and time $t = 0$.
- Each time t, take a live vertex w and check the pairs $\{w, w'\}$ for neutral w':
  - if $\{w, w'\} \in E$, make w' live.
  - otherwise, leave it neutral.

  Then, set w dead.
- When there are no live vertices, the process terminates.
  - $C(v)$ is the set of dead vertices.

- Let Z_t be the number of w' with $\{w, w'\} \in E$ at time t, and Y_t be the number of live vertices at time t. Then,

$$\begin{aligned} Y_0 &= 1, \\ Y_t &= Y_{t-1} + Z_t - 1. \end{aligned}$$

- Since no pair $\{w, w'\}$ is ever examined twice,

$$Z_t \sim Bin[n - (t-1) - Y_{t-1}, p].$$

- Let T be the least t for which $Y_t = 0$. Then, $T = |C(v)|$.
- We recursively define Y_t for all $0 \leq t \leq n$.

Lemma *For all t,*

$$Y_t \sim Bin[n-1, 1-(1-p)^t] + 1 - t.$$

Proof.

- Let $N_t = n - t - Y_t$ be the number of neutral vertices at time t.
- Note that
$$N_t \sim Bin[n-1, (1-p)^t].$$
- Then,
$$\begin{aligned} Y_t &= n - 1 - N_t + 1 - t \\ &\sim Bin[n-1, 1-(1-p)^t] + 1 - t. \end{aligned}$$

$\square$

- Set $p = c/n$.
- For fixed c,
  - Y_t^*, Z_t^*, T^*, H^* : Poisson branching process with mean c
  - Y_t, Z_t, T, H : random graph process with $G(n, \frac{c}{n})$
- For any history $(z_1, \ldots, z_t)$,

$$\Pr[H^* = (z_1, \ldots, z_t)] = \prod_{i=1}^{t} \Pr[Z^* = z_i],$$

where Z^* is Poisson with mean c while

$$\Pr[H = (z_1, \ldots, z_t)] = \prod_{i=1}^{t} \Pr[Z_i = z_i],$$

where $Z_i \sim Bin[n - 1 - z_1 - \cdots - z_{i-1}, c/n]$.

- For $m = m(n) = n + o(n^{1/4})$ and $z = o(n^{1/4})$,

$$\Pr[Bin[m, c/n] = z] = \binom{m}{z}(\frac{c}{n})^z(1-\frac{c}{n})^{m-z} = (1+o(n^{-1/2}))\frac{e^{-c}c^z}{z!}$$

(uniformly).

- Hence, for $H = (z_1, \ldots, z_t)$ with $\sum_{i=1}^{t} z_i = o(n^{1/4})$,

$$\Pr[H = (z_1, \ldots, z_t)] = (1 + o(n^{-1/4}))\Pr[H^* = (z_1, \ldots, z_t)]$$

(uniformly), and so

$$\Pr[T = t] = (1 + o(n^{-1/4}))\Pr[T^* = t],$$

for $t = o(n^{1/4})$.

Theorem *For $c < 1$, $G(n, \frac{c}{n})$ almost always has components all of which have size $O(\ln n)$.*

Proof.

- Since $Y_t \sim Bin[n-1, 1-(1-p)^t] + 1 - t$ and $1-(1-p)^t \leq tp$,

$$\begin{aligned} \Pr[T > t] &\leq \Pr[Y_t > 0] \\ &= \Pr[Bin[n-1, 1-(1-p)^t] \geq t] \\ &\leq \Pr[Bin[n, tc/n] \geq t]. \end{aligned}$$

- By (generalized) Chernoff bound,

$$\begin{aligned} \Pr[T > t] &\leq \Pr[Bin[n, tc/n] \geq t] \\ &\leq e^{-\frac{(1-c)^2 t^2}{2ct} + \frac{(1-c)^3 t^3}{2c^3 t^3}} \\ &\leq c_1 e^{-c_2 t} \end{aligned}$$

  for some constants $c_1, c_2 > 0$.

- Choose c_3 satisfying $c_2 c_3 > 1$. Then,

$$\Pr[T > c_3 \ln n] \leq c_1 e^{-c_2 c_3 \ln n} = c_1 n^{-c_2 c_3} = o(n^{-1}).$$

- Since there are n choices for initial vertex v,

$$\Pr[\exists v \text{ such that } |C(v)| > c_3 \ln n] \leq n \cdot o(n^{-1}) = o(1).$$

$\square$

Theorem *For $c > 1$, $G(n, \frac{c}{n})$ almost always has a giant component of size $\sim (1 - y)n$ and all other components of size $O(\ln n)$.*

Proof.

- Let $t_0 = K \ln n$ for a large constant K.
- First, we prove the following fact.

Claim. *Let $\varepsilon, \delta > 0$ be arbitrarily small. Then,*

$$y - \varepsilon \leq \Pr[T \leq t_0] \leq y + \varepsilon,$$

and

$$1 - y - \varepsilon \leq \Pr[(1 - \delta)(1 - y)n < T < (1 + \delta)(1 - y)n] \leq 1 - y + \varepsilon,$$

for sufficiently large n.

Proof of Claim.

- Since $\Pr[T = t] = (1 + o(n^{-1/4}))\Pr[T^* = t]$ (uniformly) for $t \leq t_0$ and $\sum_{t=1}^{\infty} \Pr[T^* = t] = y$, there is $N_1 > 0$ such that

$$y - \varepsilon \leq \Pr[T \leq t_0] \leq y + \varepsilon$$

for $n \geq N_1$.

- Note that $Y_t \sim Bin[n-1, 1-(1-p)^t] + 1 - t$.
- Let $X_t \sim Bin[n-1, 1-(1-p)^t]$.
- For $t = (1+\delta)(1-y)n = \alpha n$,

$$\Pr[T \geq \alpha n] \leq \Pr[Y_{\alpha n} \geq 0] = \Pr[X_{\alpha n} \geq \alpha n - 1].$$

- From $(1-x)^y = e^{-xy+O(yx^2)}$,

$$1-(1-p)^{\alpha n} = 1-(1-\frac{c}{n})^{\alpha n} = 1-e^{-c\alpha+O(\frac{1}{n})}.$$

- Since $\alpha > 1-e^{-c\alpha}$ for $\alpha > 1-y$, by Chernoff bound,

$$\begin{aligned} \Pr[T \geq \alpha n] &\leq \Pr[X_{\alpha n} \geq \alpha n - 1] \\ &\leq \exp(-\frac{((\alpha - 1 + e^{-c\alpha - O(\frac{1}{n})})n-1)^2}{n}) \\ &\leq e^{-c_1 n}. \end{aligned}$$

  for some constant $c_1 > 0$.

- Hence, we may choose N_2 such that

$$\Pr[T \geq (1+\delta)(1-y)n] \leq \varepsilon$$

  for $n \geq N_2$.

- For $t = \alpha n$ with $\frac{\ln^2 n}{n} \leq \alpha \leq (1-\delta)(1-y)$,

$$\begin{aligned}
\Pr[Y_{\alpha n} \leq 0] &\leq \Pr[X_{\alpha n} \leq \alpha n] \\
&\leq \exp\left(-\frac{\Theta((\alpha - 1 + e^{-c\alpha - O(\frac{1}{n})})^2 n^2)}{2(1 - e^{-c\alpha - O(\frac{1}{n})})n}\right) \\
&\leq \exp\left(-\frac{c_2(\alpha - 1 + e^{-c\alpha - O(\frac{1}{n})})^2 n)}{2(1 - e^{-c\alpha - O(\frac{1}{n})})}\right)
\end{aligned}$$

for some constant $c_2 > 0$ by Chernoff bound.

- Since, for $0 \leq \alpha \leq (1-\delta)(1-y)$,

$$\alpha - 1 + e^{-c\alpha} \leq \frac{(1-\delta)(1-y) - 1 + e^{-c(1-\delta)(1-y)}}{(1-\delta)(1-y)}\alpha \leq 0,$$

we may choose $c_3 > 0$ such that

$$c_2(\alpha - 1 + e^{-c\alpha - O(\frac{1}{n})})^2 \geq c_3 \alpha^2.$$

- For $\alpha \geq 0$

$$(1 - e^{-c\alpha}) \leq (1 - e^{-c\alpha})'\alpha,$$

so we may choose $c_4 > 0$ such that

$$2(1 - e^{-c\alpha - O(\frac{1}{n})}) \leq c_4 \alpha.$$

- Set $c_5 = \frac{c_3}{c_4} > 0$, then

$$\frac{c_2(\alpha - 1 + e^{-c\alpha - O(\frac{1}{n})})^2}{2(1 - e^{-c\alpha - O(\frac{1}{n})})} \geq c_5 \alpha \geq c_5 \frac{\ln^2 n}{n}.$$

- From the above,

$$\Pr[Y_{\alpha n} \leq 0] \leq e^{-c_5 K \ln n} = O(n^{-2}),$$

for sufficiently large K, and so

$$\Pr[t_0 \leq T \leq (1-\delta)(1-y)n] \leq \Pr[\bigcup_{\alpha} Y_{\alpha n} \leq 0] = O(1/n),$$

where $\frac{K \ln n}{n} \leq \alpha \leq (1-\delta)(1-y)$ in the union.

- Hence, we may choose N_3 such that

$$\Pr[t_0 \leq T \leq (1-\delta)(1-y)n] \leq \varepsilon$$

for $n \geq N_3$.

- Therefore, if we let $N = \max\{N_1, N_2, N_3\}$,

$$y - \varepsilon \leq \Pr[T \leq t_0] \leq y + \varepsilon,$$

and

$$1 - y - \varepsilon \leq \Pr[(1-\delta)(1-y)n < T < (1+\delta)(1-y)n] \leq 1 - y + \varepsilon,$$

for $n \geq N$.

$\square$

- Start with $G \sim G(n,p)$, select $v = v_1 \in G$, and compute $C(v_1)$.
- Then delete $C(v_1)$, pick $v_2 \in G - C(v_1)$, and iterate.
- Note that, at each stage, the remaining graph has distribution $G(m,p)$ where m is the number of vertices.
- Let $\varepsilon, \delta > 0$ be arbitrarily small.
- Call a component $C(v)$

$$\begin{cases} \textit{small} & \text{if } |C(v)| \leq t_0, \\ \textit{giant} & \text{if } (1-\delta)(1-y) < |C(v)| < (1+\delta)(1-y), \\ \textit{failure} & \text{otherwise.} \end{cases}$$

- Let $s = \frac{\ln \varepsilon}{\ln(y+2\varepsilon)}$. Then,

$$(y+\varepsilon)^s < (y+\varepsilon)^{\frac{\ln \varepsilon}{\ln(y+\varepsilon)}} = e^{\ln(y+\varepsilon)\frac{\ln \varepsilon}{\ln(y+\varepsilon)}} = e^{\ln \varepsilon} = \varepsilon.$$

- Begin the procedure with the full graph and terminate it when
  - a giant component is found,
  - a failure component is found,
  - or s small components are found.
- At each stage, the number of remaining vertices is $m = n - O(\ln^2 n) \sim n$.
  - the cond. prob.'s of small, giant, and failure remain asymptotically the same.

- The prob. that the procedure terminates without a giant component is at most

$$\varepsilon + (y+\varepsilon)\varepsilon + \cdots + (y+\varepsilon)^{s-1}\varepsilon + (y+\varepsilon)^s \le s\varepsilon + \varepsilon = (s+1)\varepsilon,$$

  because $(y+\varepsilon)^s < \varepsilon$.

- Since $\varepsilon \ln \varepsilon \to 0$ as $\varepsilon \to 0$,

$$(s+1)\varepsilon = (\frac{\ln \varepsilon}{\ln(y+2\varepsilon)} + 1)\varepsilon \to 0$$

  as $\varepsilon \to 0$, so $(s+1)\varepsilon$ may be made arbitrarily small.

- Hence, we find a giant component with prob. at least $1-(s+1)\varepsilon$.

- The remaining graph has $m \sim yn$ vertices.
- Then, $G(m,p) = G(m, \frac{c}{n}) \sim G(m, \frac{cy}{m})$.
- As $cy = d < 1$, the maximum component size of the remaining graph is $O(\ln n)$.

$\square$

Homework 1: Exercises in pages 31, 40, 43, 52, 88, 108, 110, 112, 126 (Due 2/2/07)

List of Papers

Z. Furedi, Random Polytopes in the d-Dimensional Cube, Discrete Comput. Geom. 1: 315-319 (1986).

J. Kim and J. Roche, *Covering Cubes by Random Half Cubes with Applications to Binary Neural Networks*, J. Comput. Syst. Sci. 56(2): 223-252 (1998).

Radhakrishnan & Srinivasan, *Improved Bounds and Algorithms for Hypergraph 2-coloring*, Random Structures & Algorithms 16, 4-32, (2000).

N. Alon, J. Kim and J. Spencer Nearly Perfect Matchings in Regular Simple Hypergraphs. Israel J. of Math. 100 (1997), 171-187.