10 Poisson Cloning Models

Theorem (K) Supercritical region: Let $p = \frac{1+\varepsilon}{n}$ with $\varepsilon \gg n^{-1/3}$, and $1 \ll \alpha \ll (\varepsilon^3 n)^{1/2}$. Then

$$\Pr\left[|W(n,p) - \theta_{\varepsilon} n| \geq \alpha \Big(\frac{n}{\varepsilon}\Big)^{1/2} \right] \leq 2e^{-\Omega(\alpha^2)}.$$

Łuczak, 1990: With probability $1 - O((\varepsilon^3 n)^{-1/9})$,

$$|W(n,p) - \theta_\varepsilon n| \leq 0.2 n^{2/3}.$$

(Notice that $(\frac{n}{\varepsilon})^{1/2} \ll n^{2/3}$.)

Theorem (K) Subcritical region: Suppose $\lambda = 1 - \varepsilon$ with $n^{-1/3} \ll \varepsilon \ll 1$, then,

$$\Pr\left[W(n,p) \geq \frac{\log(\varepsilon^3 n) - 2.5\log\log(\varepsilon^3 n) + c}{-(\varepsilon + \log(1-\varepsilon))}\right] \leq 2e^{-\Omega(c)},$$

and

$$\Pr\left[W(n,p) \leq \frac{\log(\varepsilon^3 n) - 2.5\log\log(\varepsilon^3 n) - c}{-(\varepsilon + \log(1-\varepsilon))}\right] \leq 2e^{-e^{\Omega(c)}},$$

for a positive constant $c > 0$.

improving

$$(2-\alpha)\frac{\log(\varepsilon^3 n)}{\varepsilon^2} \leq W(n,p) \leq (2+\alpha)\frac{\log(\varepsilon^3 n)}{\varepsilon^2},$$

for $\alpha \gg \max\{\varepsilon, \log^{-1/2}(\varepsilon^3 n)\}$.

We may also define the **Poisson Cloning Models** for

Random k-uniform hypergraphs

Random k-SAT Problems

Random Directed Graphs

Similar analyses are possible using PCM and COLA for

- The k-core problem for hypergraph (Pittel-Spencer-Wormald, ...)
- Structures of the giant component:

$$\text{e.g.}\quad \#\text{ of vertices of degree } i \geq 2$$

(Łuczak, Pittel, ...)

- Strong component of the random directed graph

(Karp, ...)

- Pure literal rule for random k-SAT problems

(Broder-Frieze-Upfal, ...)

And more

• Unit clause algorithm for random k-SAT problems (Chao-Franco, ...)

• Karp-Sipser Algorithm

(Karp-Sipser, Aronson-Frieze-Pittel, ...)

• Giant Component of

$$H \cup G(n, p)$$

for a fixed graph H. (K-Spencer)

The k-core Problem

A k-core of a graph is a largest subgraph
with minimum degree at least k

(due to **Bollobás**).

Pittel, Spencer & Wormald ('96):
For random graph $G(n,p)$ and

$$\lambda_k = \min_{\rho>0} \frac{\rho}{P(\rho, k-1)},$$

where

$$P(\rho, k-1) := \Pr\Big(\mathrm{Poi}(\rho) \geq k-1\Big) = e^{-\rho} \sum_{l=k-1}^{\infty} \frac{\rho^l}{l!},$$

If $k \geq 3$,

$$\Pr\Big[G(n, \lambda/(n-1)) \text{ has a } k\text{-core}\Big] \to \begin{cases} 0 & \text{if } \lambda < \lambda_k - n^{-\delta} \\ 1 & \text{if } \lambda > \lambda_k + n^{-\delta}, \end{cases}$$

for any $\delta \in (0, 1/2)$, and

$$\Pr\Big[\text{either } \exists \text{ no } k\text{-core or } \exists\ \ k\text{-core of size} \geq (1 - n^{-\delta})\lambda_k^* n\Big]$$

$$\to 1.$$

Recall,

$$\lambda_k = \min_{\rho > 0} \frac{\rho}{P(\rho, k-1)}.$$

(improving Łuczak's result).

C. Cooper ($\geq$ '02): Simpler proof for

$$\Pr\Big[G(n,\lambda/(n-1)) \text{ has a } k\text{-core}\Big] \to \begin{cases} 0 & \text{if } \lambda < (1-\varepsilon)\lambda_k \\ 1 & \text{if } \lambda > (1+\varepsilon)\lambda_k . \end{cases}$$

(K) For Poisson Cloning Model $G_{PC}(n, p)$ and $k \geq 3$,

$$\Pr\Big[G_{PC}(n, \lambda/(n-1)) \text{ has a } k\text{-core }\Big] \to$$

$$\begin{cases} 0 & \text{if } \lambda_k - \lambda \gg n^{-1/2} \\ 1 & \text{if } \lambda - \lambda_k \gg n^{-1/2}, \end{cases}$$

and

$$\Pr\Big[\text{either } \exists \text{ no } k\text{-core or}$$

$$\exists\ k\text{-core of size} \geq (1 - (\omega(n)n)^{-1/2})\lambda_k^* n\Big] = 1 - o(1),$$

for any $\omega(n) \to \infty$. Recall,

$$\lambda_k = \min_{\rho > 0} \frac{\rho}{P(\rho, k-1)}.$$

• Poisson Cloning Model $G_{PC}(n,p)$: Definition

Take i.i.d Poisson random variables $d(v)$'s, $v \in V$, with mean
$\lambda = p(n-1)$,
then, for each $v \in V$, take $d(v)$ copies, or **clones**, of v.

If $\sum_{v \in V} d(v)$ is even,

generate a (uniform) random
perfect matching on the set of all clones
and then contract clones of the same vertex.

If $\sum_{v \in V} d(v)$ is odd, generate a perfect match excluding a clone and a loop.

• Equivalent Definition

Take a Poisson λn random variable M_λ and M_λ unlabelled clones. Then, independently label each clone by v chosen uniformly at random among all vertices $\{1, ..., n\}$.

If M is even,

generate a (uniform) random perfect matching on the set of all clones and then contract clones of the same vertex.

If M is odd, generate a perfect match excluding a clone and a loop.

.

• Or, take a Poisson λn random variable M_λ and M_λ unlabelled clones. Assume that these clones are ordered. Take the first two clones and match them. Then, label those two clones uniformly at random.

If M is even,

this will generate a (uniform) random perfect matching as well as a valid labelling.

If M is odd, ...

$G(n, p)$ vs. $G_{PC}(n, p)$

Theorem (K) If $pn = O(1)$, then there are positive constants c_1 and c_2 so that for any collection $\mathcal{G}$ of SIMPLE graphs

$$\Pr[G(n,p) \in \mathcal{G}] \geq c_1 \Pr[G_{PC}(n,p) \in \mathcal{G}]$$

and

$$\Pr[G(n,p) \in \mathcal{G}] \leq c_2 \left(\left(\Pr[G_{PC}(n,p) \in \mathcal{G}] \right)^{1/2} + e^{-n} \right).$$

In particular,

$$\Pr[G(n,p) \in \mathcal{G}] \to 0 \quad \text{iff} \quad \Pr[G_{PC}(n,p) \in \mathcal{G}] \to 0, \quad \text{and}$$

$$\Pr[G(n,p) \in \mathcal{G}] \to 1 \quad \text{iff} \quad \Pr[G_{PC}(n,p) \in \mathcal{G}] \to 1.$$

Proof. For $\lambda = p(n-1)$ and a fixed simple graph G with m edges,

$$\begin{aligned}
&\Pr[G_{PC}(n,p) = G] \\
&= \Pr[M_\lambda = 2m] \cdot (m\tfrac{2}{n}\tfrac{1}{n})((m-1)\tfrac{2}{n}\tfrac{1}{n}) \cdots (1\tfrac{2}{n}\tfrac{1}{n}) \\
&= \frac{e^{-\lambda n}(\lambda n)^{2m}}{(2m)!} \frac{2^m m!}{n^{2m}}.
\end{aligned}$$

Using

$$\begin{aligned}
(2m)! &= (1+O(\tfrac{1}{m}))(4\pi m)^{1/2}(\tfrac{2m}{e})^{2m} \\
&= (1+O(\tfrac{1}{m}))(\pi m)^{-1/2} 2^{2m} \Big((2\pi m)^{1/2}(\tfrac{m}{e})^m\Big)^2 \\
&= (1+O(\tfrac{1}{m}))(\pi m)^{-1/2} 2^{2m} (m!)^2
\end{aligned}$$

Using

$$(2m)! \quad = \quad (1 + O(\tfrac{1}{m}))(\pi m)^{-1/2} 2^{2m} (m!)^2$$

we have

$$\begin{aligned}
\Pr[G_{PC}(n,p) = G] &= \frac{e^{-\lambda n}(\lambda n)^{2m}}{(2m)!} \frac{2^m m!}{n^{2m}} \\
&= (\tfrac{\lambda}{n})^m e^{-\lambda n/2 + O(\frac{1}{m})} \frac{(\pi m)^{1/2} e^{-\lambda n/2} (\frac{\lambda n}{2})^m}{m!}
\end{aligned}$$

On the other hand, $\lambda = p(n-1)$ yields,

$$\begin{aligned}
\Pr[G(n,p) = G] &= p^m (1-p)^{\binom{n}{2} - m} \\
&= (\tfrac{\lambda}{n-1})^m e^{-\lambda n/2 + pm - p^2 n^2/4 + O(p^2 m + p^3 n^2)}.
\end{aligned}$$

Take the smallest $\ell_1, \ell_2 > 0$ such that, if $m \leq \lambda n/2 - \ell_1$ or $m \geq \lambda n/2 + \ell_2$, then

$$\frac{(2\pi\lambda n)^{1/2} e^{-\lambda n/2} (\frac{\lambda n}{2})^m}{m!} \leq \Pr[G_{PC}(n,p) \in \mathcal{G}]^{1/2} + e^{-n}.$$

It is easy to check that $\ell_1, \ell_2 \leq cn$ for a constant c independent of $\Pr[G_{PC}(n,p) \in \mathcal{G}]$. We consider

$$\begin{aligned}
\Pr[G(n,p) \in \mathcal{G}] \quad &= \sum_{\substack{G \in \mathcal{G} \\ -\ell_1 < |G| - \lambda n/2 < \ell_2}} \Pr[G(n,p) = G] \\
&\quad + \sum_{\substack{G \in \mathcal{G} \\ |G| \leq \lambda n/2 - \ell_1}} \Pr[G(n,p) = G] \\
&\quad + \sum_{\substack{G \in \mathcal{G} \\ |G| \geq \lambda n/2 + \ell_1}} \Pr[G(n,p) = G].
\end{aligned}$$

First,

$$\begin{aligned}
&\Pr[G_{PC}(n,p) \in \mathcal{G}] \\
\geq & \sum_{\substack{G \in \mathcal{G} \\ -\ell_1 < |G| - \lambda n/2 < \ell_2}} \Pr[G_{PC}(n,p) = G] \\
\geq & \quad c_1^{-1} \sum_{\substack{G \in \mathcal{G} \\ -\ell_1 < |G| - \lambda n/2 < \ell_2}} \Pr[G(n,p) = G] \Pr[G_{PC}(n,p) \in \mathcal{G}]^{1/2}
\end{aligned}$$

gives

$$\sum_{\substack{G \in \mathcal{G} \\ -\ell_1 < |G| - \lambda n/2 < \ell_2}} \Pr[G(n,p) = G] \leq c_1 \Pr[G_{PC}(n,p) \in \mathcal{G}]^{1/2}.$$

Second, by $p\binom{n}{2} = \lambda n/2$, it is easy to check that

$$\begin{aligned}
\sum_{\substack{G\in\mathcal{G}\\ |G|\le \lambda n/2-\ell_1}} \Pr[G(n,p)=G] &\le \sum_{\ell\le\lambda n/2-\ell_1} \Pr\left[Bin\left(\binom{n}{2},p\right)=\ell\right] \\
&\le c_2 \sum_{\ell\le\lambda n/2-\ell_1} \Pr[\mathrm{Poi}(\lambda n/2)=\ell] \\
&\le (c_2+o(1))(\pi\lambda n)^{1/2}\Pr[\mathrm{Poi}(\lambda n/2)=\lambda n/2-\ell_1].
\end{aligned}$$

Ex 1. $\Pr[Bin(m,p)=\ell] = O\Big(e^{O(p^2m)}\Pr[\mathrm{Poi}(pm)]=\ell]\Big)$

2a. $$\sum_{\ell\le\lambda n/2-\ell_1} \Pr[\mathrm{Poi}(\lambda n/2)=\ell]$$

$$\le (1+o(1))(\pi\lambda n)^{1/2}\Pr[\mathrm{Poi}(\lambda n/2)=\lambda n/2-\ell_1]$$

Similarly,

$$\sum_{\substack{G\in\mathcal{G}\\|G|\geq\lambda n/2+\ell_2}} \Pr[G(n,p)=G] \leq \sum_{\ell\geq\lambda n/2+\ell_2} \Pr\left[Bin\left(\binom{n}{2},p\right)=\ell\right]$$

$$\leq c_3 \sum_{\ell\geq\lambda n/2+\ell_2} \Pr[\mathrm{Poi}(\lambda n/2)=\ell]$$

Ex.

2b. $$\sum_{\ell\geq\lambda n/2+\ell_2} \Pr[\mathrm{Poi}(\lambda n/2)=\ell]$$

$$\leq (1+o(1))(\pi\lambda n)^{1/2}\Pr[\mathrm{Poi}(\lambda n/2)=\lambda n/2+\ell_2]$$

11 Cut-Off Line Lemma

For θ in the range $0 \leq \theta \leq 1$, let $\Lambda(\theta)$ be the cut-off value when $(1-\theta^2)\lambda n$ or more clones are matched for the first time. Conversely, let $M(\theta)$ be the number of matched clones until the cut-off line reaches $\theta\lambda$.

Lemma 11.1 *(Cut-off Line Lemma) For $\theta_1 < 1$ uniformly bounded below from 0 and $0 < \Delta \leq n$,*

$$\Pr\left[\max_{\theta:\theta_1 \leq \theta \leq 1} |\Lambda(\theta) - \theta\lambda| \geq \frac{\Delta}{n}\right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta_1)n}\})},$$

and

$$\Pr\left[\max_{\theta:\theta_1 \leq \theta \leq 1} |M(\theta) - (1-\theta^2)\lambda n| \geq \Delta\right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta_1)n}\})}.$$

Conditioned on $M_\lambda = M$, that is, there are M clones initially. Then,

$$E[\Lambda_1] = \Big(1 - \frac{1}{M-1}\Big)\lambda = \Big(1 - \frac{1}{M-1}\Big)\Lambda_0$$

since we took the largest number among $M-1$ i.i.d uniform random numbers from 0 to λ. Similarly, in expectation,

$$E[\Lambda_{i+1}|\Lambda_i] = \Big(1 - \frac{1}{M-2i-1}\Big)\Lambda_i$$

Precisely,

$$\Lambda_{i+1} = (1 - T_i)\Lambda_i = \lambda \prod_{j=0}^{i} \Big(1 - T_j\Big),$$

where T_i are mutually independent and

$$T_i = \text{min. of } M - 2i - 1 \text{ uniform random numbers in } [0,1],$$

i.e,

$$\Pr[T_i \geq t] = (1-t)^{M-2i-1} \approx e^{-(M-2i-1)t}.$$

- Why $M(\theta) = (1-\theta^2)\lambda n$?

At $\Lambda = \theta\lambda$, or simply at $\theta\lambda$, (that is, when the cut-off line reaches $\theta\lambda$),

$$\Delta(\theta\lambda) \approx \left(1 - \frac{1}{\lambda n - M(\theta) - 1}\right)\theta\lambda - \theta\lambda$$

implies

$$\Delta\theta \approx \frac{-\theta}{\lambda n - M(\theta) - 1}.$$

Clearly, $\Delta N(\theta) = 2$. Hence,

$$\frac{\Delta M}{\Delta\theta} \approx \frac{-2(\lambda n - M(\theta) - 1)}{\theta}$$

Lemma 11.2 *Given $M_\lambda = M$, let T_j's be mutually independent, $j = M, M-1, ..., M-2\ell$ with $N - 2\ell \gg 1$. Then, denoting $\theta_i = (1 - 2i/M)^{1/2}$, we have, for $\varepsilon \le 0.1$,*

$$\Pr\Big[\max_{i:1\le i\le \ell}\Big|\prod_{\substack{j=M\\2|(M-j)}}^{M-2i}\Big(1-T_j\Big)-\theta_i\Big|\ge\varepsilon\Big]\le 10e^{-\frac{1+o(1)}{7}\min\{\varepsilon\theta_\ell^2 M,\ \frac{\varepsilon^2\theta_\ell^2 M}{1-\theta_\ell}\}}.$$

In particular, if $\theta_\ell = \Omega(1)$, then

$$\Pr\Big[\max_{i:1\le i\le \ell}\Big|\prod_{\substack{j=M\\2|(M-j)}}^{M-2i}\Big(1-T_j\Big)-\theta_i\Big|\ge\varepsilon\Big]\le 2e^{-\Omega(\min\{\varepsilon M,\ \frac{\varepsilon^2 M}{1-\theta_\ell}\})}.$$

Recall

$$T_i = \text{min. of } M - 2i - 1 \text{ uniform random numbers in } [0,1],$$

i.e,

$$\Pr[T_i \ge t] = (1-t)^{M-2i-1} \approx e^{-(M-2i-1)t}.$$

Proof. Proof of Lemma 11.2 As

$$\prod_{\substack{j=M \\ 2|(M-j)}}^{\theta_i^2 M} (1 - T_j) = \exp\Big(\sum_{\substack{j=M \\ 2|(M-j)}}^{\theta_i^2 M} \log(1 - T_j)\Big),$$

we show a high concentration for $\log(1 - T_j)$. Since

$$\Pr[\exists\ j,\ T_j \geq 1/2] \leq \sum_{j=M}^{\theta_\ell^2 M} 2^{-j+1} \leq 2^{-\theta_\ell^2 M+2},$$

and

$$-x - x^2 \leq \log(1 - x) \leq -x \quad \forall x : 0 \leq x \leq 1/2,$$

we have, with probability at least $1 - 2^{-\theta_\ell^k M+2}$,

$$-T_j - T_j^2 \leq \log(1 - T_j) \leq -T_j, \quad \text{for all } j.$$

Thus, for $S_j = T_j - T_j^2$, and

$$T_i^* := \sum_{\substack{j=M \\ 2|(M-j)}}^{\theta_i^2 M} T_j, \text{ and } S_i^* := \sum_{\substack{j=N \\ 2|(M-j)}}^{\theta_i^2 M} S_j,$$

it is enough to show that both of $E[S_i^*]$ and $E[T_i^*]$ are very close to θ_i and S_i^*, T_i^* are highly concentrated near their means. That is, it is enough to show that

$$\Pr[\max_i |S_i^* - E[S_i^*]| \geq \varepsilon] \leq 4e^{-\frac{1+o(1)}{6}\min\{\frac{\varepsilon^2\,\theta_\ell^2 M}{1-\theta_\ell}, \varepsilon\theta_\ell^2 M\}},$$

and

$$\Pr[\max_i |T_i^* - E[T_i^*]| \geq \varepsilon] \leq 4e^{-\frac{1+o(1)}{6}\min\{\frac{\varepsilon^2\,\theta_\ell^2 M}{1-\theta_\ell}, \varepsilon\theta_\ell^2 M\}}.$$

12 The k-core Problem

A k-core of a graph is a largest subgraph
with minimum degree at least k

(due to **Bollobás**).

Pittel, Spencer & Wormald ('96):
For random graph $G(n, p)$ and

$$\lambda_k = \min_{\rho > 0} \frac{\rho}{Q(\rho, k-1)},$$

where

$$Q(\rho, k-1) := \Pr\Big(\mathrm{Poi}(\rho) \geq k-1\Big) = e^{-\rho} \sum_{l=k-1}^{\infty} \frac{\rho^l}{l!},$$

If $k \geq 3$,

$$\Pr\left[G(n, \lambda/(n-1)) \text{ has a } k\text{-core}\,\right] \to \begin{cases} 0 & \text{if } \lambda < \lambda_{_k} - n^{-\delta} \\ 1 & \text{if } \lambda > \lambda_{_k} + n^{-\delta}, \end{cases}$$

for any $\delta \in (0, 1/2)$, and

$$\Pr\left[\text{either } \exists \text{ no } k\text{-core or } \exists\ \ k\text{-core of size} \geq (1 - n^{-\delta})\lambda_{_k}^* n\right]$$

$$\to 1.$$

Recall,

$$\lambda_k = \min_{\rho > 0} \frac{\rho}{Q(\rho, k-1)}.$$

(improving Łuczak's result).

C. Cooper ($\geq$ '02): Simpler proof for

$$\Pr\Big[G(n,\lambda/(n-1)) \text{ has a } k\text{-core}\Big] \to \begin{cases} 0 & \text{if } \lambda < (1-\varepsilon)\lambda_k \\ 1 & \text{if } \lambda > (1+\varepsilon)\lambda_k\,. \end{cases}$$

(K) For Poisson Cloning Model $G_{PC}(n,p)$ and $k \geq 3$,

$$\Pr\Big[G_{PC}(n, \lambda/(n-1)) \text{ has a } k\text{-core}\Big] \to$$

$$\begin{cases} 0 & \text{if } \lambda_k - \lambda \gg n^{-1/2} \\ 1 & \text{if } \lambda - \lambda_k \gg n^{-1/2}, \end{cases}$$

and

$$\Pr\Big[\text{either } \exists \text{ no } k\text{-core or}$$

$$\exists\ k\text{-core of size} \geq (1 - (\omega(n)n)^{-1/2})\lambda_k^* n\Big] = 1 - o(1),$$

for any $\omega(n) \to \infty$. Recall,

$$\lambda_k = \min_{\rho > 0} \frac{\rho}{Q(\rho, k-1)}.$$

We will be

$\boxed{\textbf{solving the problem as well as generating } G_{PC}(n,p)\textbf{.}}$

(Recall $p = \lambda/(n-1)$).

At step 0,

A vertex v is light if $d(v) < k$,
or the number of v-clones is less than k.

It is heavy, otherwise.

Take a light clone w and
then choose the largest clone excluding w.

We will be

$\boxed{\textbf{solving the problem as well as generating } G_{PC}(n,p)\textbf{.}}$

In general, at step i,

A vertex v is light if
the number of unmatched v-clones is less than k.

It is heavy, otherwise.

Take a unmatched light clone w and
then choose the largest unmatched clone except w.

• **Parameters**

Recall

$$M(\theta)\text{= the number of matched clones at } \theta\lambda.$$

For $v \in V$ and $0 \leq \theta \leq 1$, we define

$$d_v(\theta) = \text{the number of } v\text{-clones less than } \theta\lambda.$$

Let $L(\theta)$ be the number of light clones at $\theta\lambda$, and the number $H(\theta)$ of heavy clones at $\theta\lambda$ is denoted by

$$H(\theta) = \sum_{v \in V} d_v(\theta) 1(d_v(\theta) \geq k).$$

If the number of light clones has been positive until $\theta\lambda$, then all clones counted in $H(\theta)$ is not matched until $\Lambda = \theta\lambda$, and hence

$$L(\theta) = M_\lambda - M(\theta) - H(\theta).$$

Thus, the maximum θ such that $L(\theta) = 0$ is the same as the maximum θ such that $M_\lambda - M(\theta) - H(\theta) = 0$.

Lemma 12.1 *Let*

$$Q(\rho, k) = \Pr[\mathrm{Poi}(\rho) \geq k].$$

Then

$$\Pr\left[\max_{\theta:\theta_1 \leq \theta \leq 1} \left| H(\theta) - Q(\theta\lambda, k-1)\theta\lambda n \right| \geq \Delta\right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta_1)n}\})}.$$

Proof. First,

$$E[d_v(\theta)1(d_v(\theta) \geq k)] = e^{-\theta\lambda} \sum_{\ell \geq k} \ell \frac{(\theta\lambda)^\ell}{\ell!} = \theta\lambda e^{-\theta\lambda} \sum_{\ell \geq k} \frac{(\theta\lambda)^{(\ell-1)}}{(\ell-1)!}$$

implies that

$$E[H(\theta)] = Q(\theta\lambda, k-1)\theta\lambda n.$$

Applying a generalized Chernoff bound, we obtain the desired inequality.

Corollary 12.2

$$\Pr\left[\max_{\theta:\theta_1 \le \theta \le 1} \left| M_\lambda - M(\theta) - H(\theta) - \Big(\theta - Q(\theta\lambda, k-1)\Big)\theta\lambda n \right| \ge \Delta\right]$$

$$\le 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta_1)n}\})}.$$

Notice that, assuming $\theta > 0$, $\theta = Q(\theta\lambda, k-1)$ if and only if

$$\lambda = \frac{\theta\lambda}{Q(\theta\lambda, k-1)}.$$

This equation has a solution θ in the range $0 < \theta \le 1$ if and only if

$$\lambda \ge \min_{\rho > 0} \frac{\rho}{Q(\rho, k-1)}.$$

Notice that

$$\begin{aligned}\frac{d}{d\rho}\frac{\rho}{Q(\rho,k-1)} &= \frac{Q(\rho,k-1)-\rho P(\rho,k-2)}{Q^2(\rho,k-1)} \\ &= \frac{Q(\rho,k)-(k-2)P(\rho,k-1)}{Q^2(\rho,k-1)},\end{aligned}$$

and $Q(\rho,k)-(k-2)P(\rho,k-1) = -(k-2+O(\rho))P(\rho,k-1)$ as $\rho\to 0$ and $Q(\rho,k)-(k-2)P(\rho,k-1)\to 1$ as $\rho\to\infty$. Let $\rho_{\min}$ be the minimum ρ satisfying

$$Q(\rho,k)-(k-2)P(\rho,k-1)=0.$$

Let $\rho_{\min}$ be the minimum ρ satisfying

$$Q(\rho, k) - (k-2)P(\rho, k-1) = 0.$$

For $\rho < k$, we know that

$$\begin{aligned} Q(\rho, k) &= \frac{\rho^{k-1}}{(k-1)!} e^{-\rho} \sum_{\ell \geq k} \frac{(k-1)!\rho^{\ell-k+1}}{\ell!} \\ &< P(\rho, k-1) \sum_{\ell \geq k} \frac{\rho^{\ell-k+1}}{k^{\ell-k+1}} = \frac{\rho P(\rho, k-1)}{k-\rho}, \end{aligned}$$

in particular, $\rho \leq k-2$ yields $Q(\rho, k) < (k-2)P(\rho, k-1)$. Thus, $\rho_{\min} > k-2$ and for $\rho \geq \rho_{\min}$,

$$\begin{aligned} \frac{d}{d\rho}\Big(Q(\rho, k) - (k-2)P(\rho, k-1)\Big) &= (k-1)P(\rho, k-1) - (k-2)P(\rho, k-2) \\ &= (\rho - k + 2)P(\rho, k-2) > 0. \end{aligned}$$

All together, we have

$$\frac{d}{d\rho}\frac{\rho}{Q(\rho,k-1)} < 0, \quad \text{for } \rho < \rho_{\min},$$

and

$$\frac{d}{d\rho}\frac{\rho}{Q(\rho,k-1)} > 0, \quad \text{for } \rho > \rho_{\min}.$$

Theorem 12.3 *Let θ_T be the minimum θ such that $L(\theta) = 0$ for the first time, and θ_λ be the largest solution ≤ 1 of the equation*

$$\theta - Q(\theta\lambda, k-1) = 0.$$

Then

$$\Pr[|\theta_T - \theta_\lambda| \geq \sigma] = 2e^{-\Omega(\sigma^2 n)}.$$

Proof. Notice that θ_λ is the largest $\theta < 1$ such that

$$\frac{\theta\lambda}{Q(\theta\lambda, k-1)} = \lambda.$$

It is easy to check that

$$\theta - Q(\theta\lambda, k-1) = \Omega(\theta - \theta_\lambda) \text{ for } \theta > \theta_\lambda,$$

and there is a constant $\theta_1 > 0$

$$\theta - Q(\theta\lambda, k-1) = -\Omega(\theta_\lambda - \theta) \text{ for } \theta_\lambda - \theta_1 \leq \theta < \theta_\lambda.$$

Thus,

$$
\begin{aligned}
\Pr[\theta_T \geq \theta_\varepsilon + \sigma] &= \Pr[\min_{\theta \geq \theta_\lambda + \sigma} L(\theta) = 0] \\
&\leq \Pr[\min_{\theta \geq \theta_\lambda + \sigma} M_\lambda - M(\theta) - H(\theta) \leq 0] \\
&\leq 2e^{-\Omega(\sigma^2 n)},
\end{aligned}
$$

and

$$
\begin{aligned}
\Pr[\theta_T < \theta_\varepsilon - \sigma] &= \Pr[\min_{\theta \geq \theta_\lambda - \sigma} L(\theta) > 0] \\
&\leq \Pr[\min_{\theta \geq \theta_\lambda - \sigma} M_\lambda - M(\theta) - H(\theta) > 0] \\
&\leq 2e^{-\Omega(\min\{\theta_1^2 n, \sigma^2 n\})} = 2e^{-\Omega(\sigma^2 n)}.
\end{aligned}
$$

$\square$

Corollary 12.4

$$\Pr\left[\left||V_C| - Q(\theta_\lambda \lambda, k)n\right| \geq \sigma n\right] = 2e^{-\Omega(\sigma^2 n)}.$$

13 The Emergence Of the Giant Component

Theorem 13.1 *Supercritical Phase: Let* $\lambda := \lambda(n,p) = 1+\varepsilon$ *with* $n^{-1/3} \ll \varepsilon \ll 1$, $\mu := (1-\theta_\varepsilon)\lambda$ *and* $1 \ll \alpha \ll (\varepsilon^3 n)^{1/2}$. *Then, with probability* $1 - e^{-\Omega(\alpha^2)}$, $G_{PC}(n,p)$ *may be decomposed by three vertex disjoint graphs* C, S *and* G, *where* C *is connected and*

$$\theta_\varepsilon n - \alpha(n/\varepsilon)^{1/2} \le |C| \le \theta_\varepsilon n + \alpha(n/\varepsilon)^{1/2},$$

and $|S| \le \frac{\alpha^2}{\varepsilon^2}$, *and* G *has the same distribution as* $G_{PC}(n^*, p^*)$ *for some* n^* *and* p^* *satisfying*

$$(1-\theta_\varepsilon)n - \alpha(n/\varepsilon)^{1/2} \le n^* \le (1-\theta_\varepsilon)n + \alpha(n/\varepsilon)^{1/2},$$

and

$$\mu - \alpha(\varepsilon n)^{-1/2} \le \lambda(n^*, p^*) \le \mu + \alpha(\varepsilon n)^{-1/2}.$$

Subcritical Phase: *Suppose* $\lambda := \lambda(n,p) = 1-\varepsilon$ *with* $n^{-1/3} \ll \varepsilon \ll 1$. *Then, the size* $\ell_1^{PC}(n,p)$ *of the largest component of* $G_{PC}(n,p)$ *satisfies*

$$\Pr\left[\ell_1^{PC}(n,p) \geq \frac{\log(\varepsilon^3 n) - 2.5\log\log(\varepsilon^3 n) + c}{-(\varepsilon + \log(1-\varepsilon))}\right] \leq 2e^{-\Omega(c)},$$

and

$$\Pr\left[\ell_1^{PC}(n,p) \leq \frac{\log(\varepsilon^3 n) - 2.5\log\log(\varepsilon^3 n) - c}{-(\varepsilon + \log(1-\varepsilon))}\right] \leq 2e^{-e^{\Omega(c)}},$$

for any positive constant $c > 0$.

Inside Window: Suppose $\lambda := \lambda(n,p) = 1+\varepsilon$ *with* $|\varepsilon| = O(n^{1/3})$. *Then, whp,*

$$\ell_1^{PC}(n,p) = \Theta(n^{2/3}).$$

(All constants in $\Omega(\cdot)$*'s do not depend on any of* ε, α *and* c.*)*

Proof. (Supercritical Region) Let

$$\theta_1 = \frac{\alpha^2}{\theta_\varepsilon^2 n}, \quad \theta_2 = \theta_\varepsilon - \alpha(\theta_\varepsilon n)^{-1/2},$$

and $\theta_3 = \theta_\varepsilon + \alpha(\theta_\varepsilon n)^{-1/2}$. (Recall θ_ε is the larger solution of the equation $1 - \theta - e^{-(1+\varepsilon)\theta} = 0$.)

Let $H(\theta)$ be the number of clones of vertices that have no clones larger than or equal to $\theta\lambda$, i.e.,

$$H(\theta) = \sum_{v \in V} d_v(\theta) 1(d_v - d_v(\theta) = 0),$$

and let

$$B(\theta) = M_\lambda - M(\theta) - H(\theta).$$

Denoted by $F(\theta)$ is the number of clones activated by free steps. Then the number $A(\theta)$ of active clones at $\theta\lambda$ satisfies

$$B(\theta) \le A(\theta) \le B(\theta) + F(\theta).$$

We will show that each of the following events occur with probability $1 - e^{-\Omega(\alpha^2)}$:

(i) For $\rho = \alpha^2(\theta_\varepsilon n)^{-1}$, we have $F(1-\theta_1) \le M(V_\rho)$.

(ii) For θ in the range $\theta_1 \le \theta \le \theta_2$, all $B(1-\theta)$ are positive.

(iii) For some θ between θ_2 and θ_3, $\mathring{A}(1-\theta) = 0$.

The proof basically follows from

(1)
$$\begin{aligned} E[B(1-\theta)] &\approx \lambda n - (1-(1-\theta)^2)\lambda n - (1-\theta)\lambda e^{-\theta\lambda} n \\ &= (1-\theta-e^{-\theta\lambda})(1-\theta)\lambda n. \end{aligned}$$

(2) The random variable $B(1-\theta) = M_\lambda - M(1-\theta) - H(1-\theta)$ is highly concentrated near $(1-\theta-e^{-\theta\lambda})(1-\theta)\lambda n$.

14 Random Graph vs. Random Regular graph

An attempt to study RRG by means of RG or vice versa:

G_d =random d-regular graph,
$G = G(n, (1 - o(1))d/n), H = G(n, o(d/n))$ independent random graphs

Conjecture For $\log n \ll d \leq n/2$, there is a coupling on (G_d, G, H) such that

$$\Pr[G \subseteq G_d \subseteq G \cup H] = 1 - o(1).$$

If true, k-connectivity, Hamiltonicity, independence number, (list)chromatic number, the second largest eigenvalue, ... (cf: Copper, Frieze & Reed
Krivelevich, Sudakov, Vu & Wormald)

Partial Result:

Theorem (K & Vu) For $d = n^{\delta}$ with $0 < \delta < 1/3$, there is a coupling on (G_d, G, H) and a constant $c = c(\delta) > 0$ such that

$$\Pr[G \subseteq G_d, \Delta(G_d \setminus (G \cup H)) < c] = 1 - o(1),$$

where $\Delta(F)$ is the maximum degree of F.

Generating random d-regular graphs

- List all regular graphs and choose one randomly
- Use the configuration model

Recall,

$$\Pr[\text{Simple}] \sim \exp\Big(-\frac{d^2-1}{4}\Big).$$

• **An algorithm of Steger and Wormald to generate a SIMPLE perfect matching in the configuration model**

(1) We pick pairs of clones one by one.

(2) We never pick an edge which creates a loop. Namely, we never pick pairs of clones of the same vertex.

(3) Assume that a bunch of edges are picked. In the next step, we only pick a pair of clones that does not create a parallel edge.

Such a pair that does not create a loop is called suitable.

• **An algorithm of Steger and Wormald to generate a SIMPLE perfect matching in the configuration model**

(I) Start with a set M of nd clones (nd even) partitioned into n groups of size d.

(II) Choose two unmatched clones u, v uniformly at random. If the pair u, v is suitable, match the pair. If not, u, v remain unmatched. Repeat until no suitable pair exists.

(III) If all clones are matched, output it. Otherwise return to step (I).

• An algorithm of Steger and Wormald to generate a SIMPLE perfect matching in the configuration model

(I) Start with a set M of nd clones (nd even) partitioned into n groups of size d.

(II) Choose two unmatched clones u, v uniformly at random. If the pair u, v is suitable, match the pair. If not, u, v remain unmatched. Repeat until no suitable pair exists.

(III) If all clones are matched, output it. Otherwise the algorithm fails.

Q1: What is the probability of 'fail'?

Q2: If it succeeds, is it uniform (among all simple perfect matching)?

An algorithm of Steger and Wormald:

Does it generate the (uniform) random regular graph?

An algorithm of Steger and Wormald:

Does it generate the (uniform) random regular graph?

No! But, almost.

Theorem (Steger & Wormald) If $d = o(n^{1/28})$, then for every d-regular graph G on n vertices

$$\Pr[\text{the algorithm yields } G] = (1 + o(1))p_u.$$

Theorem (K & Vu)

If $d = o(n^{1/3}/\log^{1/2} n)$, then for every d-regular graph G on n vertices

$$\Pr[\text{the algorithm yields } G] = (1 + o(1))p_u.$$

G_d =random d-regular graph,

$G = G(n, (1 - o(1))d/n)$ independent random graphs

Theorem For $d = n^{\delta}$ with $0 < \delta < 1/3$, there is a coupling on (G_d, G) such that

$$\Pr[G \subseteq G_d] = 1 - o(1).$$

Proof idea. We keep choosing uniform random edge {u,v} of K_n with REPITITION and regard u, v as their clones $(u, i), (v, j)$, where (u, i) and (v, j) are chosen uniformly at random among all unmatched clones of u and v, respectively.

G_d =random d-regular graph,

$G = G(n, (1 - o(1))d/n)$ independent random graphs

Theorem For $d = n^\delta$ with $0 < \delta < 1/3$, there is a coupling on (G_d, G) such that

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This does not work!

Why?