14 Random Graph vs. Random Regular graph

An attempt to study RRG by means of RG or vice versa:

G_d =random d-regular graph,
$G = G(n, (1 - o(1))d/n), H = G(n, o(d/n))$ independent random graphs

Conjecture For $\log n \ll d \leq n/2$, there is a coupling on (G_d, G, H) such that

$$\Pr[G \subseteq G_d \subseteq G \cup H] = 1 - o(1).$$

If true, k-connectivity, Hamiltonicity, independence number, (list)chromatic number, the second largest eigenvalue, ... (cf: Copper, Frieze & Reed
Krivelevich, Sudakov, Vu & Wormald)

Partial Result:

Theorem (K & Vu) For $d = n^\delta$ with $0 < \delta < 1/3$, there is a coupling on (G_d, G, H) and a constant $c = c(\delta) > 0$ such that

$$\Pr[G \subseteq G_d, \Delta(G_d \setminus (G \cup H)) < c] = 1 - o(1),$$

where $\Delta(F)$ is the maximum degree of F.

Generating random d-regular graphs

• An algorithm of Steger and Wormald to generate a SIMPLE perfect matching in the configuration model

(1) We pick pairs of clones one by one.

(2) We never pick an edge which creates a loop. Namely, we never pick pairs of clones of the same vertex.

(3) Assume that a bunch of edges are picked. In the next step, we only pick a pair of clones that does not create a parallel edge.

Such a pair that does not create a loop is called suitable.

• An algorithm of Steger and Wormald to generate a SIMPLE perfect matching in the configuration model

(I) Start with a set V_d of nd clones (nd even) partitioned into n groups of size d.

(II) Choose two unmatched clones u, v uniformly at random. If the pair u, v is suitable, match the pair. If not, u, v remain unmatched. Repeat until no suitable pair exists.

(III) If all clones are matched, output it. Otherwise, the algorithm fails.

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Q1: What is the probability of 'fail'?

Q2: If it succeeds, is it uniform (among all perfect matchings that yield SIMPLE d-regular graphs)?

An algorithm of Steger and Wormald:

Does it generate the (uniform) random regular graph?

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No! But, almost.

Theorem (Steger & Wormald) If $d = o(n^{1/28})$, then for every d-regular graph G on n vertices

$$\Pr[\text{the algorithm yields } G] = (1 + o(1))p_u.$$

Theorem (K & Vu)

If $d = o(n^{1/3}/\log^{1/2} n)$, then for every d-regular graph G on n vertices

$$\Pr[\text{the algorithm yields } G] = (1 + o(1))p_u.$$

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Theorem For $d = n^\delta$ with $0 < \delta < 1/3$, there is a coupling on (G_d, G) such that

$$\Pr[G \subseteq G_d] = 1 - o(1).$$

Proof idea. We keep choosing uniform random edge $e_t := \{u_t, v_t\}$ of K_n with REPITITION and regard u, v as their clones $(u, i), (v, j)$, where (u, i) and (v, j) are chosen uniformly at random among all unmatched clones of u and v, respectively. Match (u, i) and (v, j).

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This does not work!

Why?

For a suitable edge $\{(u,i),(v,j)\}$ at the conclusion of step $t-1$, the probability that $\{(u,i),(v,j)\}$ is added in step t is

$$\binom{n}{2}^{-1} \frac{1}{d_{t-1}(u)d_{t-1}(v)},$$

where $d_{t-1}(u)$ is the number of unmatched u-clones at the conclusion of step $t-1$.

Rejection

We reject $e_t = \{u_t, v_t\}$ with probability

$$1 - \frac{d_{t-1}(u_t)d_{t-1}(v_t)}{\Lambda_{t-1}^2},$$

where $\Lambda_{t-1} = \max_u d_{t-1}(u)$.

Theorem 14.1 *Let* $\log n \ll d \ll n^{1/3}/\log^2 n$ *and* $\varphi(d)$ *be any function with* $(d \log n)^{1/2} \leq \varphi(d) \ll d$. *Then there is a joint distribution, or coupling, on* (G, G_d, GUH) *such that*

(1) The distribution of G_d *is that of the uniform random d-regular graph on n vertices. The distributions of G and H are those of the Erdős-Rényi random graphs with edge probability* $\frac{d}{n}(1 - O((\frac{\log n}{d})^{1/3}))$ *and* $O(\frac{\varphi(d)}{n})$, *respectively. Moreover, G is a random subgraph of* GUH *with edge probability* $p_G = 1 - O(\frac{\varphi(d)}{d} + (\frac{\log n}{d})^{1/3})$.

(2)

$$\Pr[H \subseteq G_d] = 1 - o(1).$$

(3)

$$\Pr\left[\Delta(G_d \setminus (G \cup H)) \leq \frac{(1 + o(1)) \log n}{\log(\varphi(d)/\log n)}\right] = 1 - o(1),$$

especially, for $d = n^{\nu}$ *with* $0 < \nu < 1/3$ *and* $\varphi(d) = d/\log\log n$*,*

$$\Pr\left[\Delta(G_d \setminus (GUH)) \le \frac{1+o(1)}{\nu}\right] = 1 - o(1),$$

where $\Delta(F)$ *is the maximum degree of* F*.*

Proof idea. For $t = 1, ..., dn/2 + 12\varphi(d)n$, take i.i.d uniform random edges $e_t := \{u_t, v_t\}$ of K_n. We also take i.i.d uniform random numbers a_t and let $\xi = (\log n/d)^{1/3}$. Suppose we have $G(t-1), M(t-1), H(t-1)$ after step $t-1$. Then, for $t_1 := d/2 - (300/\xi^2)n\log n$,

$$G(t) = G(t-1) \cup \{e_t\}1(a_t \ge \xi)1(t \le t_1),$$

$$H(t) = H(t-1) \cup \{e_t\}1(e_t \notin G(t)),$$

and, for the rejection probability

$$p_t(e_t) := 1 - \frac{d_{t-1}(u_t)d_{t-1}(v_t)}{\Lambda_{t-1}^2},$$

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$$p_t(e_t) := 1 - \frac{d_{t-1}(u_t)d_{t-1}(v_t)}{\Lambda_{t-1}^2},$$

$$M(t) = M(t-1) \cup \{(u_t, i), (v_t, j)\}1(e_t \text{ is suitable})1(a_t \geq p_t(e_t)),$$

where $(u_t, i), (v_t, j)$ are chosen uniformly at random among all unmatched u_t-clones and v_t-clones, respectively.

Clearly, $\widetilde{M}(t) \subseteq G(t) \cup H(t)$, and, provided $\max_e p_t(e) \leq \xi$ for all $t = 1, ..., t_1$,

$$G(t) \subseteq \widetilde{M}(t).$$

Thus, it is enough to show that

(1) With high probability, $\max_e p_t(e) \leq \xi$ for all $t = 1, ..., t_1$, and

(2) With high probability, the matching $M(dn/2 + 12\varphi(d)n)$ is almost perfect, i.e., it leave only few clones unmatched.

(Recall $t = 1, ...dn/2 + 12\varphi(d)n$.)