1 Poisson Cloning Models

Theorem (K) Supercritical region: Let $p = \frac{1+\varepsilon}{n}$ with $\varepsilon \gg n^{-1/3}$, and $1 \ll \alpha \ll (\varepsilon^3 n)^{1/2}$. Then

$$\Pr\Big[\,|W(n,p) - \theta_\varepsilon n| \geq \alpha\Big(\frac{n}{\varepsilon}\Big)^{1/2}\Big] \leq 2e^{-\Omega(\alpha^2)}.$$

Łuczak, 1990: With probability $1 - O((\varepsilon^3 n)^{-1/9})$,

$$|W(n,p) - \theta_\varepsilon n| \leq 0.2n^{2/3}.$$

(Notice that $(\frac{n}{\varepsilon})^{1/2} \ll n^{2/3}$.)

Theorem (K) Subcritical region: Suppose $\lambda = 1 - \varepsilon$ with $n^{-1/3} \ll \varepsilon \ll 1$, then,

$$\Pr\Big[\, W(n,p) \geq \frac{\log(\varepsilon^3 n) - 2.5\log\log(\varepsilon^3 n) + c}{-(\varepsilon + \log(1-\varepsilon))} \Big] \leq 2e^{-\Omega(c)},$$

and

$$\Pr\Big[\, W(n,p) \leq \frac{\log(\varepsilon^3 n) - 2.5\log\log(\varepsilon^3 n) - c}{-(\varepsilon + \log(1-\varepsilon))} \Big] \leq 2e^{-e^{\Omega(c)}},$$

for a positive constant $c > 0$.

improving

$$(2-\alpha)\frac{\log(\varepsilon^3 n)}{\varepsilon^2} \leq W(n,p) \leq (2+\alpha)\frac{\log(\varepsilon^3 n)}{\varepsilon^2},$$

for $\alpha \gg \max\{\varepsilon, \log^{-1/2}(\varepsilon^3 n)\}$.

We may also define the **Poisson Cloning Models** for

Random k-uniform hypergraphs

Random k-SAT Problems

Random Directed Graphs

Similar analyses are possible using PCM and COLA for

• The k-core problem for hypergraph
(Pittel-Spencer-Wormald, ...)

• Structures of the giant component:

$$\text{e.g.} \quad \# \text{ of vertices of degree } i \geq 2$$

(Łuczak, Pittel, ...)

• Strong component of the random directed graph

(Karp, ...)

• Pure literal rule for random k-SAT problems
(Broder-Frieze-Upfal, ...)

And more

• Unit clause algorithm for random k-SAT problems (Chao-Franco, ...)

• Karp-Sipser Algorithm

(Karp-Sipser, Aronson-Frieze-Pittel, ...)

• Giant Component of

$$H \cup G(n,p)$$

for a fixed graph H. (K-Spencer)

The k-core Problem

A k-core of a graph is a largest subgraph
with minimum degree at least k

(due to **Bollobás**).

Pittel, Spencer & Wormald ('96):
For random graph $G(n,p)$ and

$$\lambda_k = \min_{\rho>0} \frac{\rho}{P(\rho, k-1)},$$

where

$$P(\rho, k-1) := \Pr\Big(\mathrm{Poi}(\rho) \geq k-1\Big) = e^{-\rho} \sum_{l=k-1}^{\infty} \frac{\rho^l}{l!},$$

If $k \geq 3$,

$$\Pr\Big[G(n, \lambda/(n-1)) \text{ has a } k\text{-core}\Big] \to \begin{cases} 0 & \text{if } \lambda < \lambda_k - n^{-\delta} \\ 1 & \text{if } \lambda > \lambda_k + n^{-\delta}, \end{cases}$$

for any $\delta \in (0, 1/2)$, and

$$\Pr\Big[\text{either } \exists \text{ no } k\text{-core or } \exists \ k\text{-core of size} \geq (1 - n^{-\delta})\lambda_k^* n\Big]$$

$$\to 1.$$

Recall,

$$\lambda_k = \min_{\rho > 0} \frac{\rho}{P(\rho, k-1)}.$$

(improving Łuczak's result).

C. Cooper ($\geq$ '02): Simpler proof for

$$\Pr\left[G(n, \lambda/(n-1)) \text{ has a } k\text{-core}\right] \to \begin{cases} 0 & \text{if } \lambda < (1-\varepsilon)\lambda_k \\ 1 & \text{if } \lambda > (1+\varepsilon)\lambda_k. \end{cases}$$

(K) For Poisson Cloning Model $G_{PC}(n,p)$ and $k \geq 3$,

$$\Pr\left[G_{PC}(n, \lambda/(n-1)) \text{ has a } k\text{-core}\right] \to$$

$$\begin{cases} 0 & \text{if } \lambda_k - \lambda \gg n^{-1/2} \\ 1 & \text{if } \lambda - \lambda_k \gg n^{-1/2}, \end{cases}$$

and

$$\Pr\Big[\text{either } \exists \text{ no } k\text{-core or}$$

$$\exists\, k\text{-core of size} \geq (1-(\omega(n)n)^{-1/2})\lambda_k^* n\Big] = 1 - o(1),$$

for any $\omega(n) \to \infty$. Recall,

$$\lambda_k = \min_{\rho>0} \frac{\rho}{P(\rho, k-1)}.$$

We will be

solving the problem as well as generating $G_{PC}(n,p)$.

(Recall $p = \lambda/(n-1)$).

At step 0,

A vertex v is light if $d(v) < k$,
or the number of v-clones is less than k.

It is heavy, otherwise.

Take a light clone w and
then choose the largest clone except w.

We will be

solving the problem as well as generating $G_{PC}(n, p)$.

In general, at step i,

A vertex v is light if
the number of unmatched v-clones is less than k.

It is heavy, otherwise.

Take a unmatched light clone w and
then choose the largest unmatched clone except w.

• Poisson Cloning Model $G_{PC}(n,p)$: Definition

Take i.i.d Poisson random variables $d(v)$'s, $v \in V$, with mean $\lambda = p(n-1)$,
then, for each $v \in V$, take $d(v)$ copies, or **clones**, of v.

If $\sum_{v \in V} d(v)$ is even,

generate a (uniform) random
perfect matching on the set of all clones
and then contract clones of the same vertex.

If $\sum_{v \in V} d(v)$ is odd, generate a perfect match excluding a clone and a loop.

- Equivalently,

take a Poisson λn random variable M and M unlabelled clones. Then, independently label each clone by v chosen uniformly at random among all vertices $\{1, ..., n\}$.

If M is even,

generate a (uniform) random perfect matching on the set of all clones and then contract clones of the same vertex.

If M is odd, generate a perfect match excluding a clone and a loop.

• Or, take a Poisson λn random variable M_λ and M_λ unlabelled clones. Assume that these clones are ordered. Take the first two clones and match them. Then, label those two clones uniformly at random.

If M is even,

this will generate a (uniform) random perfect matching as well as a valid labelling.

If M is odd, ...

$G(n,p)$ vs. $G_{PC}(n,p)$

Theorem (K) If $pn = O(1)$, then there are positive constants c_1 and c_2 so that for any collection $\mathcal{G}$ of SIMPLE graphs

$$c_1 \Pr[G_{PC}(n,p) \in \mathcal{G}] - e^{-\Omega(pn^2)} \leq \Pr[G(n,p) \in \mathcal{G}],$$

and

$$\Pr[G(n,p) \in \mathcal{G}] \leq c_2 \Big(\Pr[G_{PC}(n,p) \in \mathcal{G}] \Big)^{1/2} + e^{-\Omega(pn^2)}.$$

In particular,

$$\Pr[G(n,p) \in \mathcal{G}] \to 0 \quad \text{iff} \quad \Pr[G_{PC}(n,p) \in \mathcal{G}] \to 0, \quad \text{and}$$

$$\Pr[G(n,p) \in \mathcal{G}] \to 1 \quad \text{iff} \quad \Pr[G_{PC}(n,p) \in \mathcal{G}] \to 1.$$

Proof. For $\lambda = p(n-1)$ and a fixed simple graph G with m edges,

$$\begin{aligned}
&\Pr[G_{PC}(n,p) = G] \\
&= \Pr[M_\lambda = 2m] \cdot (m\tfrac{2}{n}\tfrac{1}{n})((m-1)\tfrac{2}{n}\tfrac{1}{n}) \cdots (1\tfrac{2}{n}\tfrac{1}{n}) \\
&= \frac{e^{-\lambda n}(\lambda n)^{2m}}{(2m)!}\frac{2^m m!}{n^{2m}}.
\end{aligned}$$

Using

$$\begin{aligned}
(2m)! &= (1+O(\tfrac{1}{m}))(4\pi m)^{1/2}(\tfrac{2m}{e})^{2m} \\
&= (1+O(\tfrac{1}{m}))(\pi m)^{-1/2}2^{2m}\Big((2\pi m)^{1/2}(\tfrac{m}{e})^m\Big)^2 \\
&= (1+O(\tfrac{1}{m}))(\pi m)^{-1/2}2^{2m}(m!)^2
\end{aligned}$$

Using

$$(2m)! \;=\; (1+O(\tfrac{1}{m}))(\pi m)^{-1/2}2^{2m}(m!)^2$$

we have

$$\begin{aligned}\Pr[G_{PC}(n,p)=G] &= \frac{e^{-\lambda n}(\lambda n)^{2m}}{(2m)!}\frac{2^m m!}{n^{2m}}\\ &= (\tfrac{\lambda}{n})^m e^{-\lambda n/2+O(\frac{1}{m})}\frac{(\pi m)^{1/2}e^{-\lambda n/2}(\frac{\lambda n}{2})^m}{m!}\end{aligned}$$

On the other hand, $\lambda = p(n-1)$ yields,

$$\begin{aligned}\Pr[G(n,p)=G] &= p^m(1-p)^{\binom{n}{2}-m}\\ &= (\tfrac{\lambda}{n-1})^m e^{-\lambda n/2+pm-p^2n^2/4+O(p^2m+p^3n^2)}.\end{aligned}$$

Take the smallest $\ell_1, \ell_2 > 0$ such that, if $m \leq \lambda n/2 - \ell_1$ or $m \geq \lambda n/2 + \ell_2$, then

$$\frac{(2\pi\lambda n)^{1/2} e^{-\lambda n/2} (\frac{\lambda n}{2})^m}{m!} \leq \Pr[G_{PC}(n,p) \in \mathcal{G}]^{1/2} + e^{-n}.$$

It is easy to check that $\ell_1, \ell_2 \leq cn$ for a constant c independent of $\Pr[G_{PC}(n,p) \in \mathcal{G}]$. We consider

$$\begin{aligned}
\Pr[G(n,p) \in \mathcal{G}] \quad = \quad & \sum_{\substack{G \in \mathcal{G} \\ -\ell_1 < |G| - \lambda n/2 < \ell_2}} \Pr[G(n,p) = G] \\
& + \sum_{\substack{G \in \mathcal{G} \\ |G| \leq \lambda n/2 - \ell_1}} \Pr[G(n,p) = G] \\
& + \sum_{\substack{G \in \mathcal{G} \\ |G| \geq \lambda n/2 + \ell_1}} \Pr[G(n,p) = G].
\end{aligned}$$

First,

$$\begin{aligned}
&\Pr[G_{PC}(n,p) \in \mathcal{G}] \\
&\geq \sum_{\substack{G \in \mathcal{G} \\ -\ell_1 < |G| - \lambda n/2 < \ell_2}} \Pr[G_{PC}(n,p) = G] \\
&\geq c_1^{-1} \sum_{\substack{G \in \mathcal{G} \\ -\ell_1 < |G| - \lambda n/2 < \ell_2}} \Pr[G(n,p) = G] \Pr[G_{PC}(n,p) \in \mathcal{G}]^{1/2}
\end{aligned}$$

gives

$$\sum_{\substack{G \in \mathcal{G} \\ -\ell_1 < |G| - \lambda n/2 < \ell_2}} \Pr[G(n,p) = G] \leq c_1 \Pr[G_{PC}(n,p) \in \mathcal{G}]^{1/2}.$$

Second, by $p\binom{n}{2} = \lambda n/2$ it is easy to check that

$$\sum_{\substack{G\in\mathcal{G}\\ |G|\le\lambda n/2-\ell_1}} \Pr[G(n,p)=G] \le \sum_{\ell\le\lambda n/2-\ell_1} \Pr\left[Bin\Big(\binom{n}{2},p\Big)=\ell\right]$$

$$\le c_2 \sum_{\ell\le\lambda n/2-\ell_1} \Pr[\mathrm{Poi}(\lambda n/2)=\ell].$$

Ex 1. $\Pr[Bin(m,p)=\ell] = O\Big(e^{O(p^2m)}\Pr[\mathrm{Poi}(pm)]=\ell]\Big)$

2a. $$\sum_{\ell\le\lambda n/2-\ell_1} \Pr[\mathrm{Poi}(\lambda n/2)=\ell]$$

$$= (1+o(1))(\pi\lambda n)^{1/2}\Pr[\mathrm{Poi}(\lambda n/2)=\lambda n/2-\ell_1]$$

Similarly,

$$\sum_{\substack{G\in\mathcal{G}\\ |G|\geq\lambda n/2+\ell_2}} \Pr[G(n,p)=G] \leq \sum_{\ell\geq\lambda n/2+\ell_2} \Pr\left[Bin\Big(\binom{n}{2},p\Big)=\ell\right]$$

$$\leq c_3 \sum_{\ell\geq\lambda n/2+\ell_2} \Pr[\mathrm{Poi}(\lambda n)=\ell]$$

Ex.

2b. $$\sum_{\ell\geq\lambda n/2+\ell_2} \Pr[\mathrm{Poi}(\lambda n/2)=\ell]$$

$$= (1+o(1))(\pi\lambda n)^{1/2}\Pr[\mathrm{Poi}(\lambda n/2)=\lambda n/2+\ell_2]$$

2 Cut-Off Line Lemma

For θ in the range $0 \leq \theta \leq 1$, let $\Lambda(\theta)$ be the cut-off value when $(1-\theta^2)\lambda n$ or more clones are matched for the first time. Conversely, let $N(\theta)$ be the number of matched clones until the cut-off line reaches $\theta\lambda$.

Lemma 1 *(Cut-off Line Lemma) For $\theta_1 < 1$ uniformly bounded below from 0 and $0 < \Delta \leq n$,*

$$\Pr\left[\max_{\theta:\theta_1 \leq \theta \leq 1} |\Lambda(\theta) - \theta\lambda| \geq \frac{\Delta}{n}\right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta_1)n}\})},$$

and

$$\Pr\left[\max_{\theta:\theta_1 \leq \theta \leq 1} |N(\theta) - (1-\theta^2)\lambda n| \geq \Delta\right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta_1)n}\})}.$$