Matchings and Independent Sets of a Fixed Size in Regular Graphs

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Abstract

We use an entropy based method to study two graph maximization problems. We upper bound the number of matchings of fixed size ℓ in a d-regular graph on N vertices. For $\frac{2\ell}{N}$ bounded away from 0 and 1, the logarithm of the bound we obtain agrees in its leading term with the logarithm of the number of matchings of size ℓ in the graph consisting of $\frac{N}{2d}$ disjoint copies of $K_{d,d}$. This provides asymptotic evidence for a conjecture of S. Friedland *et al.*. We also obtain an analogous result for independent sets of a fixed size in regular graphs, giving asymptotic evidence for a conjecture of J. Kahn. Our bounds on the number of matchings and independent sets of a fixed size are derived from bounds on the partition function (or generating polynomial) for matchings and independent sets.

1 Introduction

Given a d-regular graph G on N vertices and a particular type of subgraph, a natural class of problems arises: "How many subgraphs of this type can G contain?" In this paper we give upper bounds on the number of partial matchings of a fixed fractional size, and on the number of independent sets

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of a fixed size, in a general d-regular graph, and we show that our bounds are asymptotically matched at the logarithmic level by the graph consisting of $\frac{N}{2d}$ disjoint copies of $K_{d,d}$. (See [2] and [4] for graph theory basics.)

Let G be a bipartite graph on N vertices with partition classes A and B. Suppose that the degree sequence of one side is given by $\{r_i\}_{i=1}^{|A|}$. It follows from the well-known theorem of Brégman concerning the permanent of 0-1 matrices [3] (see also [1]) that we can bound the number of perfect matchings in G using the following expression:

Theorem 1.1 *(Brégman) Let $\mathcal{M}_{\text{perfect}}(G)$ be the set of perfect matchings in G. Then*

$$|\mathcal{M}_{\text{perfect}}(G)| \leq \prod_{i=1}^{|A|} (r_i!)^{\frac{1}{r_i}}.$$

When $r_i = d$ for all i and $|A|$ is divisible by d, equality in the above theorem is achieved by the graph consisting of $\frac{N}{2d}$ disjoint copies of the complete bipartite graph $K_{d,d}$, so we know that among d-regular bipartite graphs on N vertices, with $2d|N$, this graph contains the greatest number of perfect matchings. (Wanless [12] has considered the case when $2d$ is not a multiple of N, obtaining lower bounds on $|\mathcal{M}_{\text{perfect}}(G)|$ and some structural results on the maximizing graphs in this case.)

Friedland *et al.* [6] propose an extension of this observation, which they call the Upper Matching Conjecture. Write $m_\ell(G)$ for the number of matchings in G of size ℓ, and write $DK_{N,d}$ for the graph consisting of $\frac{N}{2d}$ disjoint copies of $K_{d,d}$.

Conjecture 1.2 *For any N-vertex, d-regular graph G with $2d|N$ and any $0 \leq \ell \leq N/2$,*

$$m_\ell(G) \leq m_\ell(DK_{N,d}).$$

In this note we upper bound the logarithm of the number of ℓ-matchings of a regular graph and show that, at the level of the leading term, this upper bound is achieved by the disjoint union of the appropriate number of copies of $K_{d,d}$. We will use the parameterization $\alpha = \frac{2\ell}{N}$, and refer interchangeably to a matching of size ℓ or a matching whose size is an α-fraction of the maximum possible matching size. In what follows, $H(x) = -x \log x - (1-x)\log(1-x)$ is the usual binary entropy function. (All logarithms in this note are base 2.)

Theorem 1.3 *Let G be a d-regular graph on N vertices and ℓ an integer satisfying $0 \le \ell \le \frac{N}{2}$. Set $\alpha = \frac{2\ell}{N}$. The number of matchings in G of size ℓ satisfies*

$$\log(m_\ell(G)) \le \frac{N}{2}\left[\alpha \log d + H(\alpha)\right].$$

This bound is tight up to the first order term: for fixed $\alpha \in (0,1)$,

$$\log(m_\ell(DK_{N,d})) \ge \frac{N}{2}\left[\alpha \log d + 2H(\alpha) + \alpha \log\left(\frac{\alpha}{e}\right) + \Omega\left(\frac{\log d}{d}\right)\right],$$

with the constant in the Ω term depending on α.

In [7] an asymptotic variant of Conjecture 1.2 is presented. Let $\{G_k\}$ be a sequence of d-regular bipartite graphs with $|V_k|$, the number of vertices of G_k, growing to infinity, and fix $\alpha \in [0,1]$. Set

$$h_{\{G_k\}}(\alpha) = \limsup(\log m_{\ell_k}(G_k))/|V_k|$$

where the limit is over all sequences $\{\ell_k\}$ with $2\ell_k/|V_k| \to \alpha$. The Asymptotic Upper Matching Conjecture asserts that

$$h_{\{G_k\}}(\alpha) \le h_{\{kK_{d,d}\}}(\alpha)$$

where $kK_{d,d}$ is the graph consisting of k disjoint copies of $K_{d,d}$. Theorem 1.3 shows that for each fixed α, there is a constant c_α (independent of d) with $h_{\{G_k\}}(\alpha) \le h_{\{kK_{d,d}\}}(\alpha) + c_\alpha$.

We show similar results for the number of independent sets in d-regular graphs. A point of departure for our consideration of independent sets is the following result of Kahn [10]. For any graph G write $\mathcal{I}(G)$ for the set of independent sets in G and write $i_t(G)$ for the set of independent sets of size t (i.e., with t vertices).

Theorem 1.4 *(Kahn) For any N-vertex, d-regular bipartite graph G,*

$$|\mathcal{I}(G)| \le |\mathcal{I}(K_{d,d})|^{N/2d}.$$

Note that when $2d|N$, we have $|\mathcal{I}(K_{d,d})|^{N/2d} = |\mathcal{I}(DK_{N,d})|$. Kahn [10] proposes the following natural conjecture.

Conjecture 1.5 *For any N-vertex, d-regular graph G with $2d|N$ and any $0 \le t \le N/2$,*

$$i_t(G) \le i_t(DK_{N,d}).$$

We provide asymptotic evidence for this conjecture.

Theorem 1.6 *For N-vertex, d-regular G, and $0 \le t \le N/2$,*

$$i_t(G) \le \begin{cases} 2^{\frac{N}{2}\left(H\left(\frac{2t}{N}\right)+\frac{2}{d}\right)} & \text{in general} \\ 2^{\frac{N}{2}\left(H\left(\frac{2t}{N}\right)+\frac{1}{d}-\frac{\log e}{2d}\left(1-\frac{2t}{N}\right)^d\right)} & \text{if } G \text{ is bipartite} \\ 2^t\binom{\frac{N}{2}}{t} & \text{if } G \text{ has a perfect matching.} \end{cases} \quad (1)$$

On the other hand,

$$i_t(DK_{N,d}) \ge \begin{cases} \left(1-\frac{1}{c}\right)\binom{\frac{N}{2}}{t}2^{\frac{N}{2}\left(\frac{1}{d}-\frac{c}{d}\left(1-\frac{2t}{N}\right)^d\right)} & \text{for any } c>1 \\ 2^t\binom{\frac{N}{2}}{t}\prod_{k=1}^{t-1}\left(1-\frac{2kd}{N}\right) & \text{for } t \le \frac{N}{2d}. \end{cases} \quad (2)$$

If N, d and t are sequences satisfying $t = \alpha\frac{N}{2}$ for some fixed $\alpha \in (0,1)$ and G is a sequence of N-vertex, d-regular graphs, then from (1)

$$\log i_t(G) \le \begin{cases} \frac{N}{2}\left[H(\alpha)+\frac{2}{d}\right] & \text{in general} \\ \frac{N}{2}\left[H(\alpha)+\frac{1}{d}\right] & \text{if } G \text{ is bipartite,} \end{cases}$$

whereas if $N = \omega(d\log d)$ and $d = \omega(1)$ then taking $c=2$ in the first bound of (2) and using Stirling's formula to analyze the behavior of $\binom{N/2}{\alpha N/2}$, we obtain the near matching lower bound

$$\log i_t(DK_{N,d}) \ge \frac{N}{2}\left[H(\alpha)+\frac{1}{d}(1+o(1))\right].$$

If $N = o\left(d/(1-\alpha)^d\right)$ and G is bipartite, then the gap between our bounds on $i_t(G)$ and $i_t(DK_{N,d})$ is just a multiplicative factor of $O(\sqrt{N})$; indeed, in this case (taking any $c=\omega(1)$) we obtain from the first bound of (2) that

$$i_t(DK_{N,d}) \ge (1-o(1))\binom{\frac{N}{2}}{t}2^{\frac{N}{2}\left(H(\alpha)+\frac{1}{d}\right)}.$$

For smaller sets, whose sizes scale with N/d rather than N, the final bounds in (1) and (2) come into play. Specifically, for any N, t and d

$$i_t(DK_{N,d}) \geq \begin{cases} \binom{\frac{N}{2}}{t} 2^{t(1+o(1))} & \text{if } t = o\left(\frac{N}{d}\right) \\ (1+o(1))\binom{\frac{N}{2}}{t} 2^t & \text{if } t = o\left(\sqrt{\frac{N}{d}}\right) \end{cases} \tag{3}$$

Note that in the latter case, for G with a perfect matching we have $i_t(G) \leq (1+o(1))i_t(DK_{N,d})$. To obtain (3) from (2) we use

$$\prod_{k=1}^{t-1}\left(1 - \frac{2kd}{N}\right) \geq \exp\left\{-\frac{4d}{N}\sum_{k=1}^{t-1} k\right\} \geq \exp\left\{-\frac{2dt(t-1)}{N}\right\}.$$

2 Counting Matchings

Given a graph G and a nonnegative real number λ, we can form weighted matchings of G by assigning each matching containing ℓ edges weight λ^ℓ. The weighted partition function, $Z_\lambda^{\text{match}}(G)$, gives the total weight of matchings. Formally,

$$Z_\lambda^{\text{match}}(G) := \sum_{m \in \mathcal{M}(G)} \lambda^{|m|} = \sum_{k=0}^{\frac{N}{2}} m_k(G)\, \lambda^k.$$

(This is often referred to as the generating function for matchings or the matching polynomial). We will prove Theorem 1.3 by showing a bound on the partition function, and then using that bound to limit the number of matchings of a particular weight (size).

Lemma 2.1 *For all d-regular graphs G, $Z_\lambda^{\text{match}}(G) \leq (1+d\lambda)^{\frac{N}{2}}$*

This lemma is easily proven in the bipartite case; the difficulty arises when we want to prove the same bound for general graphs. Indeed, if G is a bipartite graph with bipartition classes A and B, we can easily see that the right hand side above counts a superset of weighted matchings. Elements in this superset are sets of edges no two of which are adjacent to the same element of A (but with no restriction on incidences with B).

Proof of Lemma 2.1 To prove this lemma, we will use the following result of Friedgut [5], which describes a weighted version of the information theoretic Shearer's Lemma.

Theorem 2.2 *(Friedgut) Let $H = (V, E)$ be a hypergraph, and $F_1, F_2, \ldots F_r$ subsets of V such that every $v \in V$ belongs to at least t of the sets F_i. Let H_i be the projection hypergraphs: $H_i = (V, E_i)$, where $E_i = \{e \cap F_i : e \in E\}$. For each edge $e \in E$, define $e_i = e \cap F_i$, and assign each e_i a nonnegative real weight $w_i(e_i)$. Then*

$$\Big(\sum_{e\in E}\prod_{i=1}^{r} w_i(e_i)\Big)^t \leq \prod_i \sum_{e_i\in E_i} w_i(e_i)^t$$

The first step in applying this theorem is to define appropriate variables. Let $G = (V, E)$ be a d-regular graph, with its vertex set $\{v_1, v_2, \ldots, v_N\}$. We will use G to form an associated matching hypergraph, $H = (E, \mathcal{M})$, where the vertex set of the hypergraph is the edge set of G, and $\mathcal{M}$ is the sets of matchings in G. Let F_i be the set of edges incident to a vertex $v_i \in V$. Note that each edge in E is covered twice by $\bigcup_{i=1}^{N} F_i$, so we may take $t = 2$. We define the trace sets, $E_i = \{F_i \cap m : m \in \mathcal{M}\}$, as the set of possible intersections of a matching with the set of edges incident with v_i. Let $m_i = m \cap F_i$. Then for all i, assign

$$w_i(m_i) = \begin{cases} 1 & \text{if } m_i = \emptyset \\ \sqrt{\lambda} & \text{else} \end{cases}$$

With these definitions we have $\sum_{m_i\in E_i} w_i(m_i)^2 = 1 + d\lambda$, and for a fixed m, $\prod_i w_i(m_i) = \sqrt{\lambda}^{(2|m|)}$. Putting these expressions into Theorem 2.2, we have that

$$(Z_\lambda^{\text{match}}(G))^2 = \left(\sum_{m\in\mathcal{M}} \lambda^{|m|}\right)^2 \leq \prod_{i=1}^{N}(1 + d\lambda).$$

Therefore,

$$Z_\lambda^{\text{match}}(G) \leq (1 + d\lambda)^{\frac{N}{2}}.$$

□

Remark 2.1 *After the submission of this paper, L. Gurvits pointed out an alternative proof of Lemma 2.1, which applies to graphs with average degree d and actually gives a slight improvement when G does not have a perfect matching. By a result of Heilmann and Lieb [9], the roots of $Z_\lambda^{\text{match}}(G) = 0$*

are all real and negative, and so we can write $Z_\lambda^{\rm match}(G) = \prod_{i=1}^{\nu(G)}(1+\alpha_i\lambda)$ *for some positive* α_i*'s with* $\sum \alpha_i = (Z_\lambda^{\rm match}(G))'\big|_{\lambda=0} = |E(G)| = \frac{Nd}{2}$*, where* $\nu(G)$ *is the size of the largest matching of* G*. Applying the arithmetic mean - geometric mean inequality to this expression we obtain*

$$Z_\lambda^{\rm match}(G) \le \left(1+\lambda\frac{\sum\alpha_i}{\nu(G)}\right)^{\nu(G)} = \left(1+\lambda\frac{Nd}{2\nu(G)}\right)^{\nu(G)} \le (1+d\lambda)^{\frac{N}{2}}.$$

Proof of Theorem 1.3 We begin with the upper bound. We may assume $0<\ell<N/2$, since the extreme cases $\ell=0, N/2$ are obvious. For fixed ℓ, a single term of the partition function $Z_\lambda^{\rm match}(G)$ is bounded by the whole sum, and so by Lemma 2.1 we have $m_\ell(G)\lambda^\ell \le Z_\lambda^{\rm match}(G) \le (1+d\lambda)^{\frac{N}{2}}$ and

$$m_\ell(G) \le (1+d\lambda)^{\frac{N}{2}}\left(\frac{1}{\lambda}\right)^\ell. \tag{4}$$

We take

$$\lambda = \frac{\ell}{d\left(\frac{N}{2}-\ell\right)}$$

to minimize the right hand side of (4) and obtain the upper bound in Theorem 1.3 (in the case $\ell = \frac{\alpha N}{2}$):

$$\begin{aligned}\log(m_\ell(G)) &\le \log\left(\frac{\frac{N}{2}}{\frac{N}{2}-\ell}\right)^{\frac{N}{2}}\left(\frac{d(\frac{N}{2}-\ell)}{\ell}\right)^\ell \\ &= \frac{N}{2}\left(\frac{2\ell}{N}\log d + H\left(2\ell/N\right)\right) \\ &= \frac{N}{2}\left(\alpha\log d + H(\alpha)\right).\end{aligned}$$

We now turn to the lower bound. We begin by observing

$$m_\ell(DK_{N,d}) = \sum_{\substack{a_1,\dots a_{N/2d}:\\ 0\le a_i\le d,\ \sum_i a_i=\ell}} \prod_{i=1}^{N/2d}\binom{d}{a_i}^2 a_i! \tag{5}$$

Here the a_i's are the sizes of the intersections of the matching with each of the components of $DK_{N,d}$, and the term $\binom{d}{a_i}^2 a_i!$ counts the number of matchings of size a_i in a single copy of $K_{d,d}$. (The binomial term represents the choice of a_i endvertices for the matching from each partition class, and the factorial term tells us how many ways there are to pair the endvertices from the top and bottom to form a matching.)

From Stirling's formula we have that there is an absolute constant $c \geq 1$ such that for any $d \geq 1$ and $0 < a < d$,

$$\log\left(\binom{d}{a}^2 a!\right) \geq a\log d + a\log\frac{a}{d} - a\log e + 2H(a/d)d - \log cd, \tag{6}$$

and we may verify by hand that (6) holds also for $a = 0, d$. Combining (5) and (6) we see that $\log(m_\ell(DK_{N,d}))$ is bounded below by

$$\frac{N}{2}\left(\frac{2\ell}{N}\log d - \frac{2\ell}{N}\log e - \frac{\log cd}{d} + \frac{2}{N}\sum_{i=1}^{N/2d}\left(a_i\log\frac{a_i}{d} + 2H(a_i/d)d\right)\right) \tag{7}$$

for any valid sequence of a_i's. To get our lower bound in the case $\ell = \alpha\frac{N}{2}$, we consider (7) for that sequence of a_i's in which each a_i is either $\lfloor \alpha d\rfloor$ or $\lceil \alpha d\rceil$. Note that by the mean value theorem, there is a constant $c_\alpha > 0$ such that both

$$\log\frac{\lceil \alpha d\rceil}{d},\ \log\frac{\lfloor \alpha d\rfloor}{d} \geq \log\alpha - \frac{c_\alpha}{d}$$

and

$$H\left(\frac{\lceil \alpha d\rceil}{d}\right),\ H\left(\frac{\lfloor \alpha d\rfloor}{d}\right) \geq H(\alpha) - \frac{c_\alpha}{d}.$$

(Here we use

$$\left|\frac{\lceil \alpha d\rceil}{d} - \alpha\right|,\ \left|\frac{\lfloor \alpha d\rfloor}{d} - \alpha\right| \leq \frac{1}{d}$$

and $\alpha \neq 0, 1$.) Putting these bounds into (7) we obtain

$$\log(m_\ell(DK_{N,d})) \geq \frac{N}{2}\left(\alpha\log d + 2H(\alpha) + \alpha\log\left(\frac{\alpha}{e}\right) + \Omega\left(\frac{\log d}{d}\right)\right),$$

with the constant in the Ω term depending on α. □

3 Counting Independent Sets

In this section we prove the various assertions of Theorem 1.6. We begin with the second bound in (1). We use a result from [8], which states that for any $\lambda > 0$ and any d-regular N-vertex bipartite graph G, the weighted independent set partition function satisfies

$$Z_\lambda^{\text{ind}}(G) := \sum_{I \in \mathcal{I}(G)} \lambda^{|I|} \leq \left(2(1+\lambda)^d - 1\right)^{\frac{N}{2d}}. \tag{8}$$

Choose λ so that $\frac{\lambda N}{2(1+\lambda)} = t$. Noting that $i_t(G)\lambda^{\frac{\lambda N}{2(1+\lambda)}}$ is the contribution to $Z_\lambda^{\text{ind}}(G)$ from independent sets of size t we have

$$\begin{aligned} i_t(G) &\leq \frac{Z_\lambda^{\text{ind}}(G)}{\lambda^{\frac{\lambda N}{2(1+\lambda)}}} \\ &\leq \frac{\left(2(1+\lambda)^d - 1\right)^{\frac{N}{2d}}}{\lambda^{\frac{\lambda N}{2(1+\lambda)}}} \\ &= 2^{\frac{N}{2d}} \left(\frac{1+\lambda}{\lambda^{\frac{\lambda}{1+\lambda}}}\right)^{N/2} \left(1 - \frac{1}{2(1+\lambda)^d - 1}\right)^{\frac{N}{2d}} \\ &= 2^{H\left(\frac{\lambda}{1+\lambda}\right)\frac{N}{2} + \frac{N}{2d}} e^{-\frac{N}{4d(1+\lambda)^d}} \\ &= 2^{H\left(\frac{2t}{N}\right)\frac{N}{2} + \frac{N}{2d} - \frac{N \log e}{4d}\left(1 - \frac{2t}{N}\right)^d}. \end{aligned} \tag{9}$$

We use (8) to make the critical substitution in (9).

To obtain the first bound in (1) we need the following analog of (8) for G not necessarily bipartite:

$$Z_\lambda^{\text{ind}}(G) \leq 2^{\frac{N}{d}} (1+\lambda)^{\frac{N}{2}}. \tag{10}$$

From (10) we easily obtain the claimed bound, following the steps of the derivation of the second bound in (1) from (8). We prove (10) by using a more general result on graph homomorphisms. For graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ set

$$Hom(G, H) = \{f : V_1 \to V_2 \ : \ \{u, v\} \in E_1 \Rightarrow \{f(u), f(v)\} \in E_2\}.$$

That is, $Hom(G, H)$ is the set of graph homomorphisms from G to H. Fix a total order $\prec$ on $V(G)$. For each $v \in V(G)$, write $P_\prec(v)$ for $\{w \in V(G) : \{w, v\} \in E(G), w \prec v\}$ and $p_\prec(v)$ for $|P_\prec(v)|$. The following natural generalization of a theorem of J. Kahn is due to D. Galvin (see [11] for a proof).

Theorem 3.1 *For any d-regular and N-vertex graph G (not necessarily bipartite) and any total order $\prec$ on $V(G)$,*

$$|Hom(G,H)| \leq \prod_{v\in V(G)} |Hom(K_{p_\prec(v),p_\prec(v)},H)|^{\frac{1}{d}}.$$

If G is bipartite with bipartition classes $\mathcal{E}$ and $\mathcal{O}$ and $\prec$ satisfies $u \prec v$ for all $u \in \mathcal{E}, v \in \mathcal{O}$ then Theorem 3.1 reduces to the main result of [8].

To prove (10), we first note that (by continuity) it is enough to prove the result for λ rational. Let C be an integer such that $C\lambda$ is also an integer, and let H_C be the graph which consists of an independent set of size $C\lambda$ and a complete looped graph on C vertices, with a complete bipartite graph joining the two. As described in [8] we have, for any graph G on N vertices,

$$|Hom(G,H_C)| = C^N Z_\lambda^{\rm ind}(G).$$

For G d-regular and N-vertex, we apply Theorem 3.1 twice to obtain

$$\begin{aligned} Z_\lambda^{\rm ind}(G) &= \frac{|Hom(G,H_C)|}{C^N} \\ &\leq \frac{\prod_{v\in V(G)} |Hom(K_{p_\prec(v),p_\prec(v)},H_C)|^{\frac{1}{d}}}{C^N} \\ &= \frac{\prod_{v\in V(G)} \left(C^{2p_\prec(v)} Z_\lambda^{\rm ind}(K_{p_\prec(v),p_\prec(v)})\right)^{\frac{1}{d}}}{C^N} \\ &\leq \frac{C^{\frac{2\sum_{v\in V(G)} p_\prec(v)}{d}} \prod_{v\in V(G)} \left(2(1+\lambda)^{p_\prec(v)}\right)^{\frac{1}{d}}}{C^N} \\ &= 2^{\frac{N}{d}} \frac{C^{\frac{2\sum_{v\in V(G)} p_\prec(v)}{d}} (1+\lambda)^{\frac{\sum_{v\in V(G)} p_\prec(v)}{d}}}{C^N}. \end{aligned}$$

Now noting that

$$\sum_{v\in V(G)} p_\prec(v) = |E(G)| = \frac{Nd}{2}$$

we obtain

$$Z_\lambda(G) \leq 2^{\frac{N}{d}}(1+\lambda)^{\frac{N}{2}},$$

as claimed.

We now turn to the third bound in (1). Fix a perfect matching of G joining a set of vertices $A \subseteq V(G)$ of size $N/2$ to the set $B := V(G) \setminus A$.

Let f be the bijection from subsets of A to subsets of B that moves the set along the chosen matching. Every independent set in G of size t is of the form $I_A \cup I_B$ where $I_A \subseteq A$, $I_B \subseteq B$, $f(A) \cap B = \emptyset$ and $|A| + |B| = t$. We therefore count all the independent sets of size t (and more) by choosing a subset of A of size t ($\binom{N/2}{t}$ choices) and a subset of this set to send to B via f (2^t choices).

To obtain the first bound in (2), we introduce a probabilistic framework and use Markov's inequality. If we divide a set of size $N/2$ into $N/2d$ blocks of size d and choose a uniform subset of size t, then the probability that this set misses a particular block is $\binom{N/2-d}{t}/\binom{N/2}{t}$. Let X be a random variable representing the number of blocks that the t-set misses. Let b_k equal the number of t-sets which miss exactly k blocks. Then $\mathbb{P}(X = k) = b_k/\binom{N/2}{t}$. Let χ_A be the indicator variable for the event A. Then

$$X = \sum_{i=0}^{\frac{N}{2d}} \chi_{\{\text{block i empty}\}}$$

and by linearity of expectation the expected number of blocks missed satisfies

$$\mu := \mathbb{E}(X) = \frac{N}{2d} \frac{\binom{\frac{N}{2}-d}{t}}{\binom{\frac{N}{2}}{t}} \leq \frac{N}{2d} \left(1 - \frac{2t}{N}\right)^d . \tag{11}$$

From Markov's inequality we have

$$\sum_{k=0}^{c\mu} \mathbb{P}(X = k) = \mathbb{P}(X \leq c\mu) \geq \left(1 - \frac{1}{c}\right).$$

We substitute the previously discussed value for $\mathbb{P}(X = k)$, yielding the inequality

$$\sum_{k=0}^{c\mu} b_k \geq \left(1 - \frac{1}{c}\right) \binom{\frac{N}{2}}{t}. \tag{12}$$

How many independent sets of size t does $DK_{N,d}$ have? To choose an independent set from $DK_{N,d}$ of size t, we first create a bipartition $\mathcal{E} \cup \mathcal{O}$ of $DK_{N,d}$ by choosing (arbitrarily) one of the bipartition classes of each of the $N/2d$ $K_{d,d}$'s of $DK_{N,d}$ to be in $\mathcal{E}$. We then choose a subset of $\mathcal{E}$ of size t. The number of subsets of $\mathcal{E}$ which have empty intersection with exactly k of the $K_{d,d}$'s that make up $DK_{N,d}$ is precisely b_k. Each of these subsets corresponds to $2^{\frac{N}{2d}-k}$ independent sets in $DK_{N,d}$. Combining this observation with (11) and (12) we obtain the first bound in (2):

$$\begin{aligned} i_t(DK_{N,d}) &= 2^{\frac{N}{2d}} \sum_{k \geq 0} 2^{-k} b_k \\ &\geq 2^{\frac{N}{2d}-c\mu} \sum_{k=0}^{c\mu} b_k \\ &\geq \left(1 - \frac{1}{c}\right) \binom{\frac{N}{2}}{t} 2^{\frac{N}{2}\left(\frac{1}{d} - \frac{c}{d}\left(1 - \frac{t}{M}\right)^d\right)}. \end{aligned}$$

Finally we turn to the second bound in (2). We obtain the claimed bound by considering all of the independent sets whose intersection with each component of $DK_{N,d}$ has size either 0 or 1:

$$i_t(DK_{N,d}) \geq (2d)^t \binom{\frac{N}{2d}}{t}.$$

After a little algebra, the right hand side above is seen to be exactly the right hand side of the second bound in (2).

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