Meander Graphs

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Abstract

We consider a meander $M = [A : B]$ to be formed from two sets A and B of n nonintersecting arcs, lying above and respectively below a horizontal line, which join to form a single closed loop. We prove that the set of meanders is connected under appropriate pairs of balanced local moves, one operating on A and the other on B. We also prove that the subset of meanders with a fixed B is connected under a suitable local move operating on an appropriately defined meandric triple in A. We provide diameter bounds under such moves, tight up to a (worst case) factor of two.

1 Introduction

A closed meander of order n is a non-self-intersecting closed curve in the plane which crosses a horizontal line at $2n$ points, up to homeomorphism in the plane. The study of meanders is said to be traceable back to Poincaré's work on differential geometry, and has subsequently arisen in different contexts such as polymer folding [6, 21, 26] and noncrossing partitions [24, 25]. Counting the number of closed meanders of order n appears reasonably difficult, and the problem remains open.

In this paper, we investigate the relationship among meanders under appropriate local moves. In this way, our results are similar to other work studying local moves which transform one Euler tour into another in any Eulerian graph [1], one Latin square into another [17], and one contingency table into another [10]. We consider a meander $M = [A : B]$ to be formed from two sets A and B of n nonintersecting arcs, lying above and respectively below a horizontal line, which join to form a single closed loop. Our results show that meanders are connected under appropriate pairs of local moves on nested arcs. In the first case, we operate on both A and B simultaneously to produce $M' = [A' : B']$ and in the second we operate twice on A to produce $M'' = [A'' : B]$. Our results also imply a tight (up to a multiplicative constant) upper bound of $2n$ on the maximum number of local moves to transform one meander into another.

The paper is organized as follows. In Section 2, we give a local move on nonintersecting arcs which is analogous to previously considered local moves on chord diagrams [22] and plane trees [16]. We prove that there exists suitable "balanced" pairs of such moves, one operating on A and the other on B, such that the result is another meander. We also show that the set of meanders is connected under such balanced pairs of local moves. In Section 3, we extend this approach to disjoint subsets of meanders, since meanders are partitioned into equivalence classes under "rotation" and "reversal." We consider the relationship among subsets of meanders with a fixed B by introducing meandric moves as a local move operation on three arcs which form a meandric triple. We keep the set of arcs below the line fixed, and operate on A by a meandric triple move, an operation which exchanges three arcs from A for another triple while preserving meandricity. Our central result is a theorem which states that meanders with a fixed set of bottom arcs B are

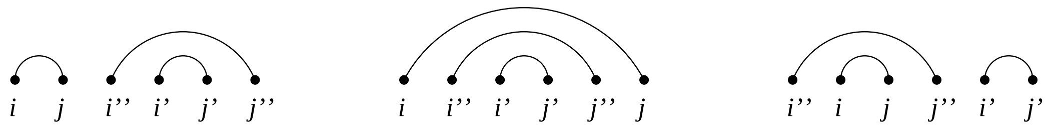

Figure 1: The three cases where an arc (i'', j'') obstructs the arcs (i, j) and (i', j') from each other. Only the relevant arcs are drawn, and the endpoints are labeled simply with the integer index.

connected under this meandric triple move. We let $\gamma(B)$ denote the graph whose vertices are all meanders with a given set B of bottom arcs and whose edges represent a meandric move. Along with a number of supporting results, we prove that

Theorem. *The graph $\gamma(B)$ is connected.*

In Section 4, we then give some characteristics, including diameter bounds, of these newly introduced meander graphs. Our connectivity results for meander graphs suggest the potential of sampling uniformly from the set of all meanders of order n, using the Markov chain Monte Carlo (MCMC) technique. Hence, we conclude in Section 5 with a brief discussion of the meander enumeration problem and uniform sampling question.

2 Balancing local moves on meanders

We consider $2n$ points $p_1, \ldots, p_{2n}$ on a fixed horizontal line l in the plane, where the points p_i occur in order of increasing index $1 \leq i \leq 2n$ from left to right along the line. Let M be a closed meander of order n which intersects l at the points p_i, and let $\mathcal{M}_n$ denote the set of all such M. See Figure 2 on page 6 for six of the eight meanders from $\mathcal{M}_3$.

We note that two combinatorial operations on the points p_i which preserve meandricity are *rotation*, that is $p_i \to p_{i-1 \pmod{2n}}$, and *reversal*, that is $p_i \to p_{2n+1-i}$. Up to equivalence under these operations, there are two distinct meanders in Figure 2 and one distinct arrangement of arcs in Figure 1 above.

We consider M to be composed of $2n$ nonintersecting arcs with endpoints $p_1, \ldots, p_{2n}$. We let (i, j) denote an arc with endpoints p_i and p_j for $i < j$, and note that each p_i is the endpoint of two arcs, one lying above the line and the other below. Let A be the set of n nonintersecting arcs lying above l, and B the set lying below. We introduce the notation $M = [A : B]$ to denote the meander M with arcs A above the points p_i and arcs B below.

Conversely, consider an arbitrary arrangement of n nonintersecting arcs with endpoints $p_1, \ldots, p_{2n}$ on the horizontal line l. Let $\mathcal{A}_n$ be the set of all possible arrangements where the n arcs lie above l, and $\mathcal{B}_n$ the set where all the arcs lie below l. Let $A \in \mathcal{A}_n$ and $B \in \mathcal{B}_n$, and consider the set of closed curves formed by the arcs of A and B, denoted $(A : B)$. We write $c(A, B) = k$ when there are k closed curves in $(A : B)$. When $c(A, B) = 1$, then the single closed curve $(A : B)$ is a meander and

$$(A : B) = [A : B] = M \text{ for some } M \in \mathcal{M}_n.$$

When $c(A, B) > 1$, then $(A : B)$ form what is known as a system of meanders.

We define a local move operation on $M \in \mathcal{M}_n$ by first considering an appropriate operation $\sigma(A)$ on a set of n nonintersecting arcs $A \in \mathcal{A}_n$. Consider two arcs $(i, j), (i', j') \in A$. If $i < i'$, then either

$$i < j < i' < j' \text{ or } i < i' < j' < j.$$

As illustrated by Figure 1, the operation $\sigma(A)$ on (i,j) and (i',j') given below will be well-defined exactly when the two arcs are unobstructed. We say that (i,j) and (i',j') are *unobstructed* if there is no third arc $(i'',j'') \in A$ where

$$i < j < i'' < i' < j' < j'' \text{ or } i < i'' < i' < j' < j'' < j \text{ or } i'' < i < j < j'' < i' < j'.$$

Let $(i,j),(i',j') \in A$ with $i < i'$. Suppose that (i,j) and (i',j') are unobstructed in A. Define $\sigma(A)$, or more explicitly $\sigma_{i,j,i',j'}(A)$, as

$$\sigma_{i,j,i',j'}(A) = \begin{cases} A \setminus \{(i,j),(i',j')\} \cup \{(i,i'),(j',j)\} & \text{if } i < i' < j' < j \\ A \setminus \{(i,j),(i',j')\} \cup \{(i,j'),(j,i')\} & \text{if } i < j < i' < j' \end{cases}.$$

This operation is analogous to previously considered local moves on chord diagrams [22] and plane trees [16]. We also consider $\sigma(B)$ for $B \in \mathcal{B}_n$, and the operation's effect on $(A : B)$.

Lemma 1. *Let $A \in \mathcal{A}_n, B \in \mathcal{B}_n$. Then $|c(A,B) - c(\sigma(A),B)| = 1$.*

Proof. Suppose $c(A,B) = k$. Let (i,j) and (i',j') be the two unobstructed arcs from $A \setminus \sigma(A)$. If (i,j) and (i',j') lie on the same curve from $(A : B)$, then $c(\sigma(A),B) = k+1$. Otherwise, they lie on two different curves and $c(\sigma(A),B) = k-1$. ☐

By symmetry, we have the same result for $\sigma(B)$. Thus, for every meander $M = [A : B]$, we have that $c(\sigma(A),B) = 2$ and $c(A,\sigma(B)) = 2$.

Theorem 1. *Let $M = [A : B] \in \mathcal{M}_n$. For every pair of unobstructed arcs in A there exists a pair of unobstructed arcs in B such that $c(\sigma(A),\sigma(B)) = 1$.*

We call the local move $\sigma(B)$ a compensatory local move for $\sigma(A)$ and refer to the pair as balanced. The result claims that there always exist a compensatory operation on B, so that we may consider the effect of transitioning between meanders $M = [A : B]$ and $M' = [\sigma(A) : \sigma(B)]$ connected by balanced pairs of local moves.

To prove Theorem 1, we introduce different notation for describing the $2n$ arcs which make up a meander or system of meanders in the plane. Let $A \in \mathcal{A}_n$ and observe that $j - i$ is odd for every arc $(i,j) \in A$. If i is odd, then we denote this arc as $i \overset{A}{\rightharpoonup} j$ and as $j \overset{A}{\rightharpoonup} i$ otherwise. Similarly, but with the reversed parity, every $(2i, 2j-1) \in B$ is written $2i \overset{B}{\rightharpoondown} 2j-1$ and every $(2j-1, 2i) \in B$ is also denoted $2i \overset{B}{\rightharpoondown} 2j-1$. In this way, a meander can be written as an ordered, alternating sequence of arcs from A and B:

$$1 \overset{A}{\rightharpoonup} 2i_1 \overset{B}{\rightharpoondown} 2j_2 - 1 \overset{A}{\rightharpoonup} 2i_2 \overset{B}{\rightharpoondown} \ldots \overset{B}{\rightharpoondown} 2j_n - 1 \overset{A}{\rightharpoonup} 2i_n \overset{B}{\rightharpoondown} 1 = 2j_1 - 1$$

for $1 \leq i_k, j_k \leq n$. Typically, we drop the A and B designation and simply write:

$$1 \rightharpoonup 2i_1 \rightharpoondown 2j_2 - 1 \rightharpoonup 2i_2 \rightharpoondown \ldots \rightharpoondown 2j_n - 1 \rightharpoonup 2i_n \rightharpoondown 1.$$

We note that a system of meanders can be written as a set of such ordered, alternating sequences of arcs from A and B.

Proof of Theorem 1. Let $M = [A : B] \in \mathcal{M}_n$. Suppose that $i \rightharpoonup j$ and $i' \rightharpoonup j'$ are two unobstructed arcs from A. It suffices to show that there exist unobstructed arcs $k \rightharpoondown l$ and $k' \rightharpoondown l'$ from B with

$$i \rightharpoonup j \ldots k \rightharpoondown l \ldots i' \rightharpoonup j' \ldots k' \rightharpoondown l' \ldots \rightharpoondown i$$

for the sequence of ordered, alternating arcs $M = [A : B]$. Let S be the set of integers which occur in the sequence of arcs $j \rightharpoondown \ldots \rightharpoondown i'$ and S' be the set occurring in $j' \rightharpoondown \ldots \rightharpoondown i$. Then S and S' are a partition of the integers $\{1, 2, \ldots, 2n\}$ and, without loss of generality, there exists $k \in S$ and $l' \in S'$ such that $|k - l'| = 1 \pmod{2n}$. Thus the arcs $k \rightharpoondown l$ and $k' \rightharpoondown l'$ are unobstructed in B. ☐

We make the previous discussion concrete by considering the following example. It will be useful at times to adopt the familial terminology from rooted trees to refer to the nesting of arcs. Consider two arcs (i, j) and (i', j') with $i < i'$. When there are no obstructing edges and $i < i' < j' < j$, then (i, j) is the parent of its child (i', j'). Otherwise, $i < j < i' < j'$ and the two unobstructed arcs are siblings. Ancestors and descendants refer to arcs with a chain of parent/child relationships. For simplicity, we consider all arcs (i, j) with $1 \leq i < j \leq 2n$ to be descendants of an (unexpressed) primordial arc $(0, 2n+1)$. We note that if (i, j) and (i', j') are unobstructed, then i and i' have opposite parity exactly when they are parent and child.

Example 1. *Let $A = \{(1,2),(3,4),(5,6),(7,8)\}$ and $B = \{(1,8),(2,3),(4,5),(6,7)\}$. Together they form the meander $M = [A : B]$ with the single closed loop $1 \rightharpoonup 2 \rightharpoondown 3 \rightharpoonup 4 \rightharpoondown 5 \rightharpoonup 6 \rightharpoondown 7 \rightharpoonup 8 \rightharpoondown 1$. Consider the local move on the unobstructed arcs $(1,2)$ and $(5,6)$ in A resulting in $\sigma(A) = A' = \{(1,6),(2,5),(3,4),(7,8)\}$. We note that the local move replaces sibling arcs $(1,2),(5,6) \in A$ with the parent/child arcs $(1,6),(2,5) \in A'$. Now $(A' : B)$ form a system of meanders with $c(A', B) = 2$ consisting of the pair of closed loops $1 \rightharpoonup 6 \rightharpoondown 7 \rightharpoonup 8 \rightharpoondown 1$ and $3 \rightharpoonup 4 \rightharpoondown 5 \rightharpoonup 2 \rightharpoondown 3$. The arcs $2 \rightharpoondown 3$ and $1 \rightharpoondown 8$ must be unobstructed in B so we have the compensatory local move $\sigma(B) = B' = \{(1,2),(3,8),(4,5),(6,7)\}$ such that $c(A', B') = 1$. Hence, we have the meander $M' = [A' : B']$ consisting of the single closed loop $1 \rightharpoonup 6 \rightharpoondown 7 \rightharpoonup 8 \rightharpoondown 3 \rightharpoonup 4 \rightharpoondown 5 \rightharpoonup 2 \rightharpoondown 1$.*

We note that the proof of Theorem 1 guarantees the existence of at least one balanced pair $\sigma(A), \sigma(B)$. In general, there may be many such compensatory operations on B. For instance, in the previous example there are three other local moves on B which also yield a meander $M' = [\sigma(A) : \sigma(B)]$. We now consider the relationship among meanders when we operate on the arcs above and below the horizontal line by pairs of balanced local moves.

Definition 1. *Let $\mathcal{G}_n$ be the graph whose vertices are $M, M' \in \mathcal{M}_n$ and whose edges connect $M = [A : B]$ and $M' = [\sigma(A) : \sigma(B)]$.*

Theorem 2. *The graph $\mathcal{G}_n$ is connected.*

Proof. Let $U_n = \{(2i-1, 2i) \mid 1 \leq i \leq n\}$ and $L_n = \{(1, 2n), (2i, 2i+1) \mid 1 \leq i < n\}$. Then $[U_n : L_n] \in \mathcal{M}_n$. For $M = [A : B] \in \mathcal{M}_n$, there exists a sequence of local moves on A such that $\sigma(\ldots\sigma(A)) = U_n$. By Theorem 1, for each local move on the upper arcs, there is a compensatory local move on the bottom. □

We note that an alternative proof of Theorem 2 follows from the connection between meanders and pairs of noncrossing partitions, see [12, 13] as well as [15, 24]. In that context, the graph $\mathcal{G}_n$ is the Hasse diagram of the induced partial order.

3 Graphing meandric triple moves

Now, we extend our approach from the full graph $\mathcal{G}_n$ to disjoint subsets of meanders. We fix $B \in \mathcal{B}_n$ and consider the graph $\gamma(B)$ of the relationship among the meanders $M = [A : B] \in \mathcal{M}_n$ under some suitable local move. Our interest in this problem arises from the equivalence of meanders under the operations of rotation and reversal, and the potential for investigating the combinatorics of $\gamma(B)$ for different representative $B \in \mathcal{B}_n$. Here, we show that the subset of meanders with fixed B is connected under a local move operation on three arcs which form a meandric triple.

Definition 2. *Let $i \rightharpoonup j$, $i' \rightharpoonup j'$, and $i'' \rightharpoonup j''$ be three arcs from $A \in \mathcal{A}_n$. The three arcs are a meandric triple if exactly two of the three pairs of arcs are unobstructed.*

Assuming no other obstructing arcs, Figure 1 on page 2 illustrates the three possible configurations for a meandric triple – which are equivalent under rotation and reversal.

Theorem 3. *Let $M = [A : B] \in \mathcal{M}_n$, and suppose there is a meandric triple in A. There exists a sequence of two local moves on the meandric triple such that $c(\sigma(\sigma(A)), B) = 1$.*

Proof. Let $i \rightharpoonup j$, $i' \rightharpoonup j'$, and $i'' \rightharpoonup j''$ be a meandric triple from A where $i \rightharpoonup j$ and $i' \rightharpoonup j'$ are unobstructed, $i \rightharpoonup j$ and $i'' \rightharpoonup j''$ are unobstructed, and

$$i \rightharpoonup j \overbrace{\ldots}^{R} i' \rightharpoonup j' \underbrace{\ldots}_{R'} i'' \rightharpoonup j'' \overbrace{\ldots}^{R''}.$$

Suppose that

$$j'' < i < j' < i' < j < i''.$$

Since the arcs (i, j) and (j', i') are unobstructed in A, the local move $\sigma_{i,j,j',i'}(A)$ is well-defined. Moreover, the arcs (i, j') and (i', j) are both unobstructed from $(j'', i'') \in \sigma(A)$. We operate on the arcs (i', j) and (j'', i'') to obtain $(j'', i'), (j, i'') \in \sigma(\sigma(A))$. Thus, $c(\sigma(\sigma(A)), B) = 1$ and we have the meander

$$i \rightharpoonup j' \underbrace{\ldots}_{R'} i'' \rightharpoonup j \overbrace{\ldots}^{R} i' \rightharpoonup j'' \overbrace{\ldots}^{R''}.$$

There are six different cases for the ordering of $p_i, p_j, \ldots$ along the horizontal line l. Under rotation, $p_i \to p_{i-1 \pmod{2n}}$, there are two distinct equivalence classes. Under reversals, $p_i \to p_{2n+1-i}$, those two classes are equivalent. $\square$

Definition 3. *Let $M = [A : B] \in \mathcal{M}_n$ and $i \rightharpoonup j$, $i' \rightharpoonup j'$, and $i'' \rightharpoonup j''$ be a meandric triple in A with $i \rightharpoonup j \ldots i' \rightharpoonup j' \ldots i'' \rightharpoonup j'' \ldots$. Define a meandric move on M, denoted $\tau(M)$, as the sequence of two local moves which results in the meander $M' = [\sigma(\sigma(A)) : B]$ where the meandric triple in A is replaced by $i \rightharpoonup j'$, $i'' \rightharpoonup j$, and $i' \rightharpoonup j''$ in $\sigma(\sigma(A))$.*

Definition 4. *For $B \in \mathcal{B}_n$, let $\gamma(B)$ be the graph whose vertices are $M = [A : B] \in \mathcal{M}_n$ and whose edges connect M and $\tau(M)$.*

Theorem 4. *For $B \in \mathcal{B}_n$, the graph $\gamma(B)$ is connected.*

The proof of Theorem 4 follows from Theorems 5 and 6, and the following related definitions.

Definition 5. *Let β_i be the arc $(i, i+1)$ for $1 \leq i < 2n$ and $\beta_{2n} = (1, 2n)$.*

Theorem 5. *For $M = [A : B] \in \mathcal{M}_n$ with $\beta_k \in B$, $\beta_{k-1 \pmod{2n}} \notin A$, there exists a meandric move $\tau(M) = [A' : B]$ such that $\beta_{k-1 \pmod{2n}} \in A'$.*

We have the following immediate dual result under reversals, $p_i \to p_{2n+1-i}$, which preserve meandric triples.

Corollary 1. *For $M = [A : B] \in \mathcal{M}_n$ with $\beta_k \in B$, $\beta_{k+1 \pmod{2n}} \notin A$, there exists a meandric move $\tau(M) = [A' : B]$ such that $\beta_{k+1 \pmod{2n}} \in A'$.*

Definition 6 makes precise the intuitive notion of contracting a bump β_i from A and the two connecting arcs in B to produce a reduced meander of order $n - 1$.

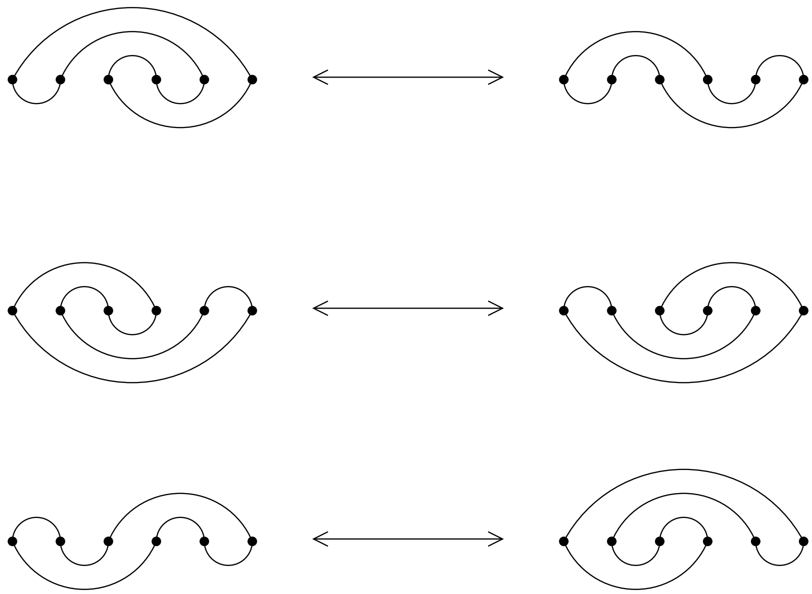

Figure 2: The six meanders from $\mathcal{M}_3$ which are connected under pairs of local moves on meandric triples. (The other two meanders, $[U_3 : L_3]$ and $[L_3 : U_3]$, have no meandric triple.) Although the endpoints are not labeled, we note the equivalences under rotations, $p_i \to p_{i-1 \pmod{2n}}$, and reversals, $p_i \to p_{2n+1-i}$.

Definition 6. *Let $M = [A : B] \in \mathcal{M}_n$. For $\beta_{2n} \in A$, define $\rho(M, 2n)$ to be the meander $[A' : B'] \in \mathcal{M}_{n-1}$ with*

$$(l, l') \in B' \quad \textit{for} \quad (1, l), (l', 2n) \in B \quad \textit{and} \quad 1 < l < l' < 2n$$

and, for $X = A, B$, with

$$(i, j) \in X' \quad \textit{for} \quad (i+1, j+1) \in X \quad \textit{and} \quad 1 < i < j < 2n.$$

For $\beta_k \in A$ with $1 \le k < 2n$, define $\rho(M, k)$ to be the meander $[A' : B'] \in \mathcal{M}_{n-1}$ with

$$\begin{cases} (l', l) \in B' & \textit{for} \quad (l', k+1), (l, k) \in B \quad \textit{and} \quad 1 \le l' < l < k < k+1 \le 2n \\ (l, l') \in B' & \textit{for} \quad (l, k), (k+1, l') \in B \quad \textit{and} \quad 1 \le l < k < k+1 < l' \le 2n \\ (l', l) \in B' & \textit{for} \quad (k, l), (k+1, l') \in B \quad \textit{and} \quad 1 \le k < k+1 < l' < l \le 2n \end{cases}$$

and, for $X = A, B$, with

$$(i, j) \in X' \textit{ whenever } \begin{cases} (i, j) \in X \quad \textit{and} \quad 1 \le i < j < k < k+1 \le 2n \\ (i, j+2) \in X \quad \textit{and} \quad 1 \le i < k < k+1 < j+2 \le 2n \\ (i+2, j+2) \in X \quad \textit{and} \quad 1 \le k < k+1 < i+2 < j+2 \le 2n. \end{cases}$$

If $\beta_k \notin A$, then $\rho(M, k)$ is not defined.

Theorem 6. *The operation $\rho : \mathcal{M}_n \times \{1, 2, \ldots, 2n\} \to \mathcal{M}_{n-1}$ is well-defined.*

Assuming Theorems 5 and 6, whose proofs are given below, we now prove Theorem 4.

Proof of Theorem 4. For $n > 3$, let $B \in \mathcal{B}_n$ and consider $M = [A : B], N = [C : B] \in \mathcal{M}_n$. We claim that M and N are connected in $\gamma(B)$ by a sequences of meandric moves.

There exists at least one $\beta_k \in B$. Let $j = k-1 \pmod{2n}$ and suppose that $\beta_j \notin A \cup C$. By Theorem 5, there exist a meandric move $\tau(M) = M' = [A' : B]$ and a meandric move $\tau(N) = N' = [C' : B]$ such that $\beta_j \in A' \cap C'$.

Observe that β_j obstructs no arcs in either A' or C'. By induction, $\rho(M', j)$ and $\rho(N', j)$ are connected by a sequence of meandric moves. Hence, there exists a sequence of meandric moves on the $n-1$ upper arcs of M' which leaves the arc $\beta_j \in A' \cap C'$ fixed and which connects M' to N'. □

Proof of Theorem 5. Under the rotational equivalence $p_i \to p_{i-1 \pmod{2n}}$, we may assume without loss of generality that $(2n-1, 2n) \in B$, $(2n-2, 2n-1) \notin A$ for $M = [A : B] \in \mathcal{M}_n$. We claim that there exists $\tau(M) = [A' : B] \in \gamma(B)$ such that $(2n-2, 2n-1) \in A'$.

The arcs

$$(l, 2n), (l', 2n-1), (l'', 2n-2) \in A \text{ with } 1 \le l < l' < l'' < 2n-2 < 2n-1 < 2n$$

are a meandric triple with

$$l \rightharpoonup 2n \rightharpoondown 2n-1 \rightharpoonup l' \ldots l'' \rightharpoonup 2n-2 \ldots .$$

After operating on this meandric triple, we have

$$l \rightharpoonup l' \ldots l'' \rightharpoonup 2n \rightharpoondown 2n-1 \rightharpoonup 2n-2 \ldots .$$

□

Proof of Theorem 6. Let $M = [A : B] \in \mathcal{M}_n$. We claim that for $\beta_k \in A$, the map $\rho(M, k)$ is well-defined.

Suppose $(1, 2n) \in A$. The arcs of A and B form the meander M:

$$l \rightharpoondown 1 \rightharpoonup 2n \rightharpoondown l' \ldots j' \rightharpoondown i \rightharpoonup j \rightharpoondown i' \ldots .$$

Then

$$l \rightharpoondown l' \ldots (j'-1) \rightharpoondown (i-1) \rightharpoonup (j-1) \rightharpoondown (i'-1) \ldots$$

is the meander $\rho(M, 2n) = [A' : B']$ since

$$A' = \{(i-1, j-1) \mid (i,j) \in A, 1 < i < j < 2n\} \in \mathcal{A}_{n-1}$$

and

$$B' = \{(i-1, j-1) \mid (i,j) \in B, 1 < i < j < 2n\} \cup \{(l, l') \mid (1, l), (l', 2n) \in B\} \in \mathcal{B}_{n-1}.$$

The case when $(k, k+1) \in A$ for $1 \le k < 2n$ is similar, although the shifting of the endpoint indices for an arc (i, j) from M depends on the ordering of i and j with respect to k. Deleting the arc $(k, k+1)$ from A, replacing the two arcs with endpoints k and $k+1$ in B with a new arc, and these shifts in indices introduce no intersections. Hence $A' \in \mathcal{A}_{n-1}$, $B' \in \mathcal{B}_{n-1}$, and $[A' : B'] = \rho(M, k) \in \mathcal{M}_{n-1}$. □

4 Some characteristics of meander graphs

We note that the proof of Theorem 4 implies that the diameter of $\gamma(B)$ is at most $2n$ for $B \in \mathcal{B}_n$. This upper bound is never achieved since for $3 \le n \le 8$, the maximum diameter of $\gamma(B)$ is $n-2$.

Example 2. *When $n = 9$, there is exactly one (nonisomorphic) pair of meanders $[A : B]$ and $[A' : B]$ whose geodesic has 8 meandric moves in $\gamma(B)$:*

$$\begin{aligned} B &= \{(1,10),(2,9),(3,8),(4,5),(6,7),(11,18),(12,17),(13,14),(15,16)\} \\ A &= \{(1,16),(2,15),(3,14),(4,13),(5,12),(6,11),(7,10),(8,9),(17,18)\} \\ A' &= \{(1,4),(2,3),(5,18),(6,17),(7,16),(8,15),(9,14),(10,13),(11,12)\} \end{aligned}$$

This is the only pair of meanders, up to rotation and reversals, whose geodesic has length greater than $n-2$ for $n = 9$. When $n = 10$, there are three nonisomorphic pairs with length 9.

We contrast this with the diameter $n-1$ of $\mathcal{G}_n$ from Definition 1. Recall that $\mathcal{G}_n$ is the graph of all meanders $M = [A : B] \in \mathcal{M}_n$ connected under balanced local moves to $M' = [\sigma(A) : \sigma(B)]$. Let $M = [A : B], M' = [A' : B'] \in \mathcal{M}_n$ be two arbitrary meanders. If $\beta_k \in A'$, there is always a local move such that $\beta_k \in \sigma(A)$ and a compensatory local move $\sigma(B)$ such that $[\sigma(A) : \sigma(B)] \in \mathcal{M}_n$. This is not the case for meandric moves; the smallest example is the following.

Example 3. *When $n = 5$, there is exactly one (nonisomorphic) pair of meanders $M = [A : B]$ and $M' = [A' : B]$ such that for every $\beta_k \in A$ there exists no $\tau(M') = [A'' : B]$ with $\beta_k \in A''$, and vice versa:*

$$\begin{aligned} B &= \{(1,10),(2,9),(3,8),(4,7),(5,6)\} \\ A &= \{(1,4),(2,3),(5,10),(6,9),(7,8)\} \\ A' &= \{(1,6),(2,5),(3,4),(7,10),(8,9)\} \end{aligned}$$

We say that such a pair of meanders is interlocking. There are no interlocking pairs when $n = 6$, eight such when $n = 7$, seven when $n = 8$, and 198 when $n = 9$.

Despite the existence of interlocking pairs, whose numbers appear to grow as some complicated function of n, we know from Theorem 4 that they must be connected in $\gamma(B)$. Hence, for $\beta_k \in A'$ and $\beta_k \notin A$, there always exists a sequence of meandric moves $\tau(\ldots\tau(M)) = [A^* : B]$ such that $(k, k+1) \in A^*$.

Theorem 7. *Let $B \in \mathcal{B}_n$ and $\beta_k \notin B$. Then there exists $M = [A : B]$ such that $\beta_k \in A$.*

Proof. The proof essentially inverts the map ρ given in Definition 6. Under rotation, we may assume that $k = 2n$. Let B' be the set of arcs with

$$\begin{aligned} (l,l') \in B' \quad &\text{for} \quad (1,l),(l',2n) \in B \quad \text{and} \quad 1 < l < l' < 2n \\ (i,j) \in B' \quad &\text{for} \quad (i+1,j+1) \in B \quad \text{and} \quad 1 < i < j < 2n. \end{aligned}$$

Then $B' \in \mathcal{B}_{n-1}$ and there exists $A' \in \mathcal{A}_{n-1}$ such that $[A' : B'] \in \mathcal{M}_{n-1}$. Let A be the set of arcs with

$$(i,j) \in A \quad \text{for} \quad (i-1,j-1) \in A' \quad \text{and} \quad 1 < i < j < 2n.$$

Then by construction $[A : B] \in \mathcal{M}_n$. ☐

Consequently, for every k such that $\beta_k \notin B$, the graph $\gamma(B)$ has a subgraph isomorphic to $\gamma(B')$, where $B' \in \mathcal{M}_{n-1}$ as in the proof of Theorem 7. By the proof of Theorem 5, we also know that every $M \in \gamma(B)$ is at most distance one from the subgraphs containing $\rho(M, k-1 \pmod{2n})$ and $\rho(M, k+1 \pmod{2n})$ for each $\beta_k \in B$.

We also have the following result. Although the theorem is an immediate corollary to Theorems 4 and 7, we give here a constructive proof as the methods illustrate some of the challenges in working with meandric triples.

Theorem 8. *Let $M = [A : B] \in \mathcal{M}_n$. For $\beta_k \notin B$, there exists a sequences of meandric moves $\tau(\ldots\tau(M)) = [A^* : B]$ such that $\beta_k \in A^*$.*

(Constructive) *Proof of Theorem 8.* We assume that k is odd. So for $m, l \neq k+1$ and $m', l' \neq k$, the meander M has the arcs

$$m \rightharpoondown k \rightharpoonup \overbrace{l \dots l'}^{R} \rightharpoonup k+1 \rightharpoondown \overbrace{m' \dots m}^{R'}.$$

Since $\beta_k \notin B$, there exists at least one arc $i \rightharpoonup j$ in the sequence of arcs R'. If $i \rightharpoonup j$, $k \rightharpoonup l$, and $k' \rightharpoonup l'$ form a meandric triple for any such arc in R', then $(k, k+1) \in \tau(M)$.

Suppose not. Consider an arc $i \rightharpoonup j$ from R' which has d arcs from R which obstruct it from forming a meandric triple with $k \rightharpoonup l$, $l' \rightharpoonup k+1$. If $d > 2$, then operating on a meandric triple from the d arcs results in a $\tau(M)$ which now has $d-2$ obstructing arcs. Hence, we need consider only when there are 1 or 2 obstructing arcs.

Although there are three cases for the ordering of the points from $k \rightharpoonup l$ and $l' \rightharpoonup k+1$ along the horizontal line, they are equivalent under rotations and reversals. Hence, we explicitly consider the case $k < k+1 < l' < l$. Note that the points $p_1, \dots, p_{2n}$ are divided into three sets by the two arcs, $S_1 = \{i \mid 1 \leq i < k, l < i \leq 2n\}$, $S_2 = \{i \mid k+1 < i < l'\}$, and $S_3 = \{i \mid l' < i < l\}$. Then i, j and the endpoints of the obstructing arcs must all be in one of the three sets. Moreover, the case when they lie in S_1 is equivalent to S_3.

Suppose there is a single obstructing arc $a \rightharpoonup b$:

$$k \rightharpoonup l \overbrace{\dots a \rightharpoonup b \dots}^{R} l' \rightharpoonup k+1 \overbrace{\dots i \rightharpoonup j \dots}^{R'}.$$

We explicitly consider the two situations when either

$$a < j < i < b < k < k+1 < l' < l \text{ or } k < k+1 < l' < b < i < j < a < l.$$

In the second case when the arcs lie in S_2, operating on M by a meandric move on $i \rightharpoonup j$, $a \rightharpoonup b$, and $k \rightharpoonup l$ followed by a move on the new meandric triple $i \rightharpoonup l$, $k \rightharpoonup b$, $l' \rightharpoonup k+1$ results in $\beta_k \in \tau(\tau(M))$. We claim the first case, when the arcs lie in S_1, results in a contradiction.

Consider $n = 4$. Then the closed loop would be

$$k \rightharpoonup l \rightharpoondown a \rightharpoonup b \rightharpoondown l' \rightharpoonup k+1 \rightharpoondown i \rightharpoonup j \rightharpoondown k.$$

However, it is not possible to have the three arcs $k+1 \rightharpoondown i$, $j \rightharpoondown k$ and $b \rightharpoondown l'$ lying below the horizontal line without intersections. Suppose $n > 4$ and there is a meander $M \in \mathcal{M}_n$ containing the arrangement of four arcs. There exists an additional arc $i'' \rightharpoonup j''$ where $|i'' - j''| = 1$. Without loss of generality, $j'' = i''+1$ and $\rho(M, i'')$ has $n-1$ arcs. Inductively, though, the arcs in $\rho(M, i'')$ corresponding to $k+1 \rightharpoondown i$, $j \rightharpoondown k$ and $b \rightharpoondown l'$ intersect.

Suppose now that there are two obstructing arcs $a \rightharpoonup b$, $a' \rightharpoonup b'$ between $i \rightharpoonup j$ and $k \rightharpoonup l$, $l' \rightharpoonup k+1$. There are two distinct orderings for a, b and a', b' along the horizontal line with respect to the other arcs. When the obstructing arcs lie in S_1, one ordering results in a contradiction like the one above while the other yields $\beta_k \in \tau(\tau(M))$. When the obstructing arcs lie in S_2, then both orderings result in a contradiction. $\square$

5 Concluding remarks

We now return to the question (briefly motivated in the introduction) of uniformly sampling from the set $\mathcal{M}_n$ of closed meanders of order n. Theorem 2 suggests a natural ergodic Markov chain, with transition probability matrix $\mathbb{P}$, on the state space $\mathcal{M}_n$: given meanders $M, M' \in \mathcal{M}_n$, we may define $\mathbb{P}(M, M')$ to be positive if M' may be obtained from $M = [A : B]$, by an (unobstructed arc) local move $\sigma(A)$ followed by a compensatory local move $\sigma(B)$. Recall the definition of these local moves from the beginning of Section 2. It is also technically convenient to assume that the self-loop probability is positive; that is, $\mathbb{P}(M,M) > 0$, for every $M \in \mathcal{M}_n$. Both these sets of probabilities will be specified (implicitly) shortly.

The fact that $\mathcal{G}_n$ is connected implies that such a Markov chain is ergodic: for every pair of states, there is a time by which the probability of visiting one state from the other is positive. The self-loop probability further guarantees aperiodicity – that a high enough power of $\mathbb{P}$ has *all* entries positive, which in turn implies that the Markov chain converges to its so-called stationary distribution on $\mathcal{M}_n$. The final fact, from the basics on finite Markov chains (or from linear algebra) that we appeal to, states that a *symmetric Markov chain* has uniform distribution as its stationary distribution. This suggests a few ways to specify the off-diagonal transition probabilities $\mathbb{P}(M,M')$, for $M \neq M'$, so as to make $\mathbb{P}$ symmetric. One fairly standard way in Markov chain Monte Carlo methods is to consider the so-called *maximum-degree random walk*: view the Markov chain, as a random walk on the graph $\mathcal{G}_n$ Let $\Delta(\mathcal{G}_n)$ denote the maximum degree of a vertex (meander) in this graph. Then we may define $\mathbb{P}(M,M') := 1/\Delta(\mathcal{G}_n)$, for every adjacent pair M, M', and define $\mathbb{P}(M,M) := 1 - \sum_{M' \neq M} \mathbb{P}(M,M')$ so as to make $\mathbb{P}$ symmetric and (row and hence column) stochastic.

There are several other ways to define $\mathbb{P}$ so that it is row and column stochastic, which is also sufficient to guarantee uniformity of stationary probabilities. However, the seemingly challenging open question we raise here is whether the above Markov chain (or an analogous one) is "rapidly mixing" on the state space of $\mathcal{M}_n$ – in the sense that, irrespective of the starting state (at time $t = 0$) the first time T_{tv}, by which the chain is within $1/4$ (say) in total variation distance of the uniform distribution, is at most polynomial in $\log|\mathcal{M}_n|$.

Note that a corresponding statement for sampling chord diagrams uniformly from the set $\mathcal{C}_n$ of all chord diagrams (with n chords) is known to be true (see e.g. [22], ...)

A second question in the same vein would be to sample uniformly from the set of meanders, with a fixed "bottom" chord diagram. Our main theorem (Theorem 4) provides, once again, a natural way to define an appropriate Markov chain, using the meandric moves, which converges to the correct (uniform) distribution; however, the rate of mixing of the chain remains open.

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