THE EXTREMAL FUNCTION FOR BIPARTITE LINKLESSLY EMBEDDABLE GRAPHS[1]

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Abstract

An embedding of a graph in 3-space is *linkless* if for every two disjoint cycles there exists an embedded ball that contains one of the cycles and is disjoint from the other. We prove that every bipartite linklessly embeddable (simple) graph on $n \geq 5$ vertices has at most $3n-10$ edges, unless it is isomorphic to the complete bipartite graph $K_{3,n-3}$.

1 Introduction

All *graphs* in this paper are finite and simple. *Paths* and *cycles* have no "repeated" vertices. An embedding of a graph in 3-space is *linkless* if for every two disjoint cycles there exists an embedded ball that contains one of the cycles and is disjoint from the other. We prove the following theorem.

Theorem 1.1. *Every bipartite linklessly embeddable graph on* $n \geq 5$ *vertices has at most* $3n - 10$ *edges, unless it is isomorphic to the complete bipartite graph* $K_{3,n-3}$.

The question whether linklessly embeddable bipartite graphs on $n \geq 5$ vertices have at most $3n-9$ edges is stated as [15, Problem 2], and Theorem 1.1 is implied by [5, Conjecture 4.5].

The following are equivalent conditions for a graph to be linklessly embeddable. A graph H is obtained from a graph G by a $Y\Delta$ *transformation*

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if H is obtained from G by deleting a vertex v of degree three and joining every pair of non-adjacent neighbors of v by an edge. Conversely, G is obtained from H by means of a ΔY *transformation* if G is obtained from H by deleting the edges of a cycle of length three ("a triangle") and adding a vertex of degree three joined to the vertices of the triangle. The *Petersen family* is the set of seven graphs obtained from the complete graph K_6 by means of $Y\Delta$ and ΔY transformations. The Petersen graph is a member of the family, and hence the name. The Petersen family is depicted in Figure 1. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. An H *minor* is a minor isomorphic to H. We denote by $\mu(G)$ the graph invariant introduced by Colin de Verdière [3]. We omit its definition, because we do not need it.

Theorem 1.2. *For a graph G the following conditions are equivalent:*

(i) *G has an embedding in 3-space such that every two disjoint cycles have even linking number.*

(ii) *G is linklessly embeddable.*

(iii) *G has an embedding in 3-space such that every cycle bounds an open disk disjoint from the embedding of G.*

(iv) *G has no minor isomorphic to a member of the Petersen family.*

(v) $\mu(G) \leq 4$.

Here (iii)⇒(ii) and (ii)⇒(i) are trivial, (i)⇒(iv) was shown by Sachs [13, 14], (iv)⇒(iii) was shown by Robertson, Seymour and the second author [12], (v)⇒(iv) was shown by Bacher and Colin de Verdière [1], and (iii)⇒(v) was shown by Lovász and Schrijver [8].

Let us now put Theorem 1.1 in perspective. For graphs that are not necessarily bipartite the correct bound on the number of edges is $4n - 10$, which follows from the following more general result of Mader [9].

Theorem 1.3. *For every integer $p = 2, 3, \ldots, 7$, a graph on $n \geq p-1$ vertices and no minor isomorphic to K_p has at most $(p-2)n - \binom{p-1}{2}$ edges.*

Theorem 1.3 is such a nice result that it raises the question of whether it can be generalized to all values of p. But there are some depressing news:

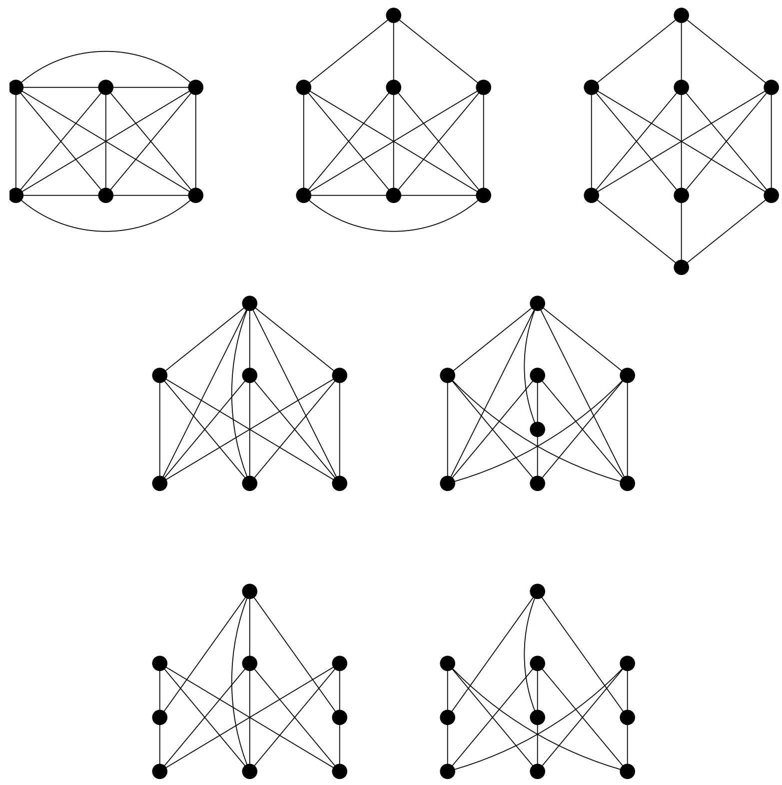

Figure 1: The Petersen family.

for large p a graph must have at least $\Omega(p\sqrt{\log p}n)$ edges in order to guarantee a K_p minor, because, as noted by several people (Kostochka [6, 7], and Fernandez de la Vega [4] based on Bollobás, Catlin and Erdös [2]), a random graph with no K_p minor may have average degree of order $p\sqrt{\log p}$. Kostochka [6, 7] and Thomason [16] proved that this is indeed the correct order of magnitude, and in a remarkable result [17] Thomason was able to determine the constant of proportionality. Thus it may seem that an effort to generalize Theorem 1.3 will be in vain, but there are still the following possibilities. The random graph examples provide only finitely many counterexamples for any given value of p. Of course, more counterexamples can be obtained by taking disjoint unions or even gluing counterexamples along small cutsets, but we know of no construction of highly connected infinite families of counterexamples. More specifically, Seymour and the second author conjecture the following.

Conjecture 1.4. *For every integer $p \geq 2$ there exists a constant $N = N(p)$ such that every $(p-2)$-connected graph on $n \geq N$ vertices and no K_p minor has at most $(p-2)n - \binom{p-1}{2}$ edges.*

In a slightly different direction the first author conjectures [10] the following.

Conjecture 1.5. *For every integer $p \geq 3$, a graph G on $n \geq p-1$ vertices and $\mu(G) \leq p-2$ has at most $(p-2)n - \binom{p-1}{2}$ edges.*

Whether Conjecture 1.5 holds is stated as [15, Problem 1]. Conjecture 1.5 is implied by [11, Conjecture 1.5].

In this paper we are concerned with bipartite graphs, but before we turn our attention to them we formulate the following conjectured generalization of Theorem 1.1.

Conjecture 1.6. *Every triangle-free linklessly embeddable graph on $n \geq 5$ vertices has at most $3n-10$ edges, unless it is isomorphic to the complete bipartite graph $K_{3,n-3}$.*

A possible approach to Conjecture 1.6 is to prove the following conjecture:

Conjecture 1.7. *Every linklessly embeddable graph on $n \geq 7$ vertices with t triangles has at most $3n - 9 + t/3$ edges.*

Let us turn to bipartite graphs now. Motivated by Theorem 1.1 and the equivalence of (ii) and (v) in Theorem 1.2 we conjecture the following.

Conjecture 1.8. *For every integer $p \geq 3$, a bipartite graph G on $n \geq 2p-3$ vertices and $\mu(G) \leq p$ has at most $(p-1)n - (p-1)^2$ edges.*

Let us remark that the bound in Conjecture 1.8, if true, is tight, because of the graphs $K_{p-1,n-p+1}$. For $p = 3$ Conjecture 1.8 follows from the fact that graphs G with $\mu(G) \leq 3$ are precisely planar graphs [3], and for $p = 4$ it follows from Theorems 1.1 and 1.2.

Let us repeat that for not necessarily bipartite graphs the bound on the number of edges for linklessly embeddable graphs and graphs with no K_6 minors coincide. Not so for bipartite graphs. We conjecture the following.

Conjecture 1.9. *For every integer $p = 2, 3, \dots, 8$, a bipartite graph on $n \geq 2p-5$ vertices and no minor isomorphic to K_p has at most $(p-2)n-(p-2)^2$ edges.*

Again, the bound in Conjecture 1.9, if true, is tight, because of the graphs $K_{p-2,n-p+2}$. For $p \leq 4$ Conjecture 1.9 is easy, and for $p = 5$ it follows from Wagner's characterization of graphs with no K_5 minor [18]. It is open for all $p = 6, 7, 8$. Conjecture 1.9 may very well hold for a few more values of p beyond 8, but it certainly does not hold for all p, because a graph with $\Omega(p\sqrt{\log p}n)$ edges and no K_p minor has a bipartite subgraph with $\Omega(p\sqrt{\log p}n)$ edges and no K_p minor.

One could speculate whether Conjectures 1.8 and 1.9 hold for triangle-free graphs, but we do not have enough evidence to formally conjecture that.

The paper is organized as follows. In the next section we introduce definitions and notation. In Section 3 we state Theorem 3.1, which implies Theorem 1.1 and prove half of it, proving some useful lemmas and disposing of vertices of degree five. In Section 4 we complete the proof of Theorem 3.1 by disposing of vertices of degree four.

2 Notation and Definitions

For positive integers $n_1, n_2, \dots, n_k$ with $k \geq 2$, we let $K_{n_1,n_2,\dots,n_k}$ denote the complete multipartite graph with k independent sets of sizes $n_1, n_2, \dots, n_k$. We let $K_{4,4}^-$ denote the graph obtained from $K_{4,4}$ by deleting an edge. We also let $K_6^{\Delta Y}$ denote the graph obtained from K_6 by performing a ΔY transformation.

For a graph G we write $V(G)$ for the vertex set of G and $E(G)$ for the edge set of G. We write $\delta(G)$ for the minimum degree of G and $\Delta(G)$ for the maximum degree of G. Suppose v is a vertex of G and S is a subset of $V(G)$. Then we write $G[S]$ for the induced subgraph of G with vertex set S and $G - S$ for the induced subgraph of G with vertex set $V(G) - S$. We write $G - v$ for $G - \{v\}$. We write $d_G(v)$, or $d(v)$ if the graph is understood from context, for the degree of v in G. We write $N_G(S)$ for the set of all vertices in $V(G) - S$ that are adjacent to some vertex in S. We write $N(S)$ if the graph is understood from context, and we write $N(v)$ for $N(\{v\})$. We use $N[v]$ to denote $N(v) \cup \{v\}$.

If G is a graph with S and T disjoint subsets of $V(G)$, we say an edge $uv \in E(G)$ is *between* S and T if $S \cap \{u, v\} \neq \emptyset$ and $T \cap \{u, v\} \neq \emptyset$. If S

consists of a single vertex v, we may talk about the edges between v and T. Given a graph G, we say that X_0 and X_1 is a *bipartition* of G if X_0 and X_1 is a partition of $V(G)$ so that all edges of G are between X_0 and X_1.

We define a *separation* of a graph G to be a pair of sets (A, B) with union $V(G)$ such that G has no edge between $A - B$ and $B - A$. The *order* of a separation (A, B) is $|A \cap B|$. We also say that a separation of order k is a k-separation. A separation (A, B) is *non-trivial* if both $A - B$ and $B - A$ are non-empty. We say that a separation (A, B) is *minimal* if there does not exist a non-trivial separation (A', B') of G with $A' \cap B' \subsetneq A \cap B$.

It is convenient for us to give the following related definition. We say a *super-separation* of a graph G is a pair of graphs (G_0, G_1) such that $V(G) \subseteq V(G_0) \cup V(G_1)$, and $E(G) \subseteq E(G_0) \cup E(G_1)$, and both G_0 and G_1 are minors of G. We say a super-separation (G_0, G_1) of G is *non-trivial* if $|V(G_0)| < |V(G)|$ and $|V(G_1)| < |V(G)|$. We say that the *order* of a super-separation (G_0, G_1) of G is $|V(G_0)| + |V(G_1)| - |V(G)|$. Finally we say a super-separation (G_0, G_1) is *bipartite* if both G_0 and G_1 are bipartite. Note that if (A, B) is a (non-trivial) separation of G of order k, then $(G[A], G[B])$ is a (non-trivial) super-separation of G of order k. Furthermore if G is bipartite then the super-separation $(G[A], G[B])$ is bipartite.

Finally, if G is a bipartite graph with bipartition X_0 and X_1 and $S \subseteq V(G)$, then we will write $\overline{G[S]}$ for the *bipartite complement* of $G[S]$. That is, $\overline{G[S]}$ is the graph on vertex set S where uv is an edge of $\overline{G[S]}$ if and only if uv is an edge between X_0 and X_1 and $uv \notin E(G)$.

3 Proof of Main Theorem: Vertices of Degree Five

By Theorem 1.2, the following theorem implies Theorem 1.1.

Theorem 3.1. *Every bipartite graph on $n \geq 5$ vertices with no K_6, $K_{1,3,3}$, $K_{4,4}^-$, or $K_6^{\Delta Y}$ minor has at most $3n - 10$ edges, unless it is isomorphic to the complete bipartite graph $K_{3,n-3}$.*

Proof. Let G be a counterexample with minimum number of vertices. Write $n := |V(G)|$, and let X_0 and X_1 be a bipartition of G.

We begin by giving a brief outline of our proof strategy. First we will show an easy lemma, and that $4 \leq \delta(G) \leq 5$. Then we show that G cannot have certain separations and super-separations of small order. It follows that G has no subgraph isomorphic to $K_{3,3}$: otherwise it either has a $K_{1,3,3}$ minor or a separation of small order. Next we show that if v is a vertex of degree four or five and x and y are neighbors of v, then x and y have several common neighbors other than v. Then it is fairly easy to show that G has no vertex v of degree five: for every pair of distinct neighbors x and y of v, let $v_{x,y}$ be a vertex other than v that is adjacent to both x and y. If all ten $v_{x,y}$ are distinct, then G has a K_6 minor. Otherwise we find a $K_{3,3}$ subgraph or another forbidden minor. In Section 4 we will deal with the case that $\delta(G) = 4$.

We begin with two easy lemmas:

Lemma 3.2. $n \geq 7$

Proof. Otherwise, we have $n \in \{5, 6\}$. Then $\lceil n/2 \rceil = 3$ and $\lfloor n/2 \rfloor = n - 3$. If G is a subgraph of $K_{3,n-3}$, then since by assumption G is not isomorphic to $K_{3,n-3}$, we have $|E(G)| < |E(K_{3,n-3})| = 3n - 9$, a contradiction. So G is not a subgraph of $K_{3,n-3}$. Then we have

$$3n - 10 < |E(G)| \leq |X_0||X_1| \leq (\lceil n/2 \rceil + 1)(\lfloor n/2 \rfloor - 1) = 4(n - 4)$$

This gives us $n > 6$, a contradiction. □

Lemma 3.3. $4 \leq \delta(G) \leq 5$

Proof. Let v be a vertex of minimum degree. Since $n \geq 6$ by Lemma 3.2, either $G - v$ is isomorphic to $K_{3,n-4}$ and $|E(G - v)| = 3(n - 1) - 9$, or $|E(G - v)| \leq 3(n - 1) - 10$. If $d(v) \leq 2$, then

$$|E(G)| = |E(G - v)| + d(v) \leq 3(n - 1) - 9 + 2 = 3n - 10$$

a contradiction. Now suppose $d(v) = 3$. If $G - v$ is not isomorphic to $K_{3,n-4}$, then similarly we have $|E(G)| \leq 3n - 10$, and we are done.

So $G - v$ is isomorphic to $K_{3,n-4}$. Without loss of generality suppose that $v \in X_0$. If $N(v) = X_1$, then G is isomorphic to $K_{3,n-3}$, a contradiction. So there exists a vertex $u \in X_1 - N(v)$. Then $|X_0 - \{v\}| = 3$, and $G[X_0 \cup \{u\} \cup N(v)]$ is isomorphic to $K_{4,4}^-$, a contradiction. So $\delta(G) = d(v) \geq 4$.

Next note that $|E(G)| = |E(G-v)| + d(v) \leq 3(n-1) - 9 + (n-1) = 4n - 13$. So $\delta(G) \leq 7$. Then $|E(G)| \leq 3(n - 1) - 9 + d(v) \leq 3n - 5$. So $\delta(G) \leq 5$, completing the proof of the lemma. □

Next we prove three lemmas on separations and super-separations of G. Observe that since $\delta(G) \geq 4$ by Lemma 3.3, if (A, B) is a non-trivial separation of G, then $(G[A], G[B])$ is a non-trivial bipartite super separation of G such that $|V(G[A])|, |V(G[B])| \geq 5$. We will frequently apply the following lemma to such a case.

Lemma 3.4. *Let (G_0, G_1) be a non-trivial bipartite super-separation of G of order k such that $|V(G_0)|, |V(G_1)| \geq 5$. Write $e := |E(G_0)| + |E(G_1)| - |E(G)|$, and let l be the number of the graphs G_0 and G_1 that are isomorphic to $K_{3,t}$ for some t. Then $3k + l - e \geq 11$.*

Proof. Suppose otherwise. So $3k + l - e - 10 \leq 0$. By the conditions of the lemma and since G is a vertex-minimum counterexample, we have:

$$\begin{aligned} |E(G)| &= |E(G_0)| + |E(G_1)| - e \leq 3(|V(G_0)| + |V(G_1)|) - 20 + l - e \\ &= 3(n+k) - 20 + l - e = 3n - 10 + (-10 + 3k + l - e) \leq 3n - 10 \end{aligned}$$

a contradiction. □

We will use the next short lemma to show that for certain separations (A, B) of G, the graph $G[A]$ is not isomorphic to $K_{3,t}$ for any t.

Lemma 3.5. *Suppose (A, B) is a separation of G and $i \in \{0, 1\}$ is such that $(A - B) \cap X_{1-i} \neq \emptyset$. Then $|A \cap X_i| \geq 4$.*

Proof. Let v be a vertex in $(A - B) \cap X_{1-i}$. Then $|A \cap X_i| \geq |N(v)| \geq 4$ since $d(v) \geq 4$ by Lemma 3.3. □

Next we show that G does not have certain separations of small order.

Lemma 3.6. *Let (A, B) be a non-trivial separation of G such that for every $i \in \{0, 1\}$, $|A \cap B \cap X_i| \leq 3$. Then $|A \cap B| = 6$ and $\Delta(G[A \cap B]) \leq 1$.*

Proof. Suppose otherwise for some separation (A, B). Note that any non-trivial separation (A', B') of G with $A' \cap B' \subsetneq A \cap B$ also violates the lemma. Thus we may assume that (A, B) is minimal.

First we show that both A and B have at least four vertices in each side of the bipartition of G. Let $v \in A - B$, and without loss of generality assume that $v \in X_0$. Then $|X_1 \cap A| \geq |N(v)| \geq 4$ since $\delta(G) \geq 4$ by Lemma 3.3. Also, since $|A \cap B \cap X_1| \leq 3$, there exists a vertex $u \in N(v) - (A \cap B)$. Then similarly $|X_0 \cap A| \geq |N(u)| \geq 4$. The same argument shows that B has at least four vertices in each side of the bipartition of G.

Now for convenience write $S := A \cap B$. Let $z \in S$ so that $d_{\overline{G[S]}}(z)$ is maximum, where $\overline{G[S]}$ is the bipartite complement of $G[S]$. Let G_A be the graph formed from $G[A]$ by adding edges between z and every vertex in $N_{\overline{G[S]}}(z)$. We can see that G_A is a minor of G by contracting some component of $G[B-A]$ to z and by the minimality of (A, B). Furthermore G_A is bipartite, has fewer vertices than G, and has at least four vertices in each side of the bipartition of G. So G_A is not isomorphic to $K_{3,t}$ for any t. Define G_B analogously, by adding edges between z and every vertex in $N_{\overline{G[S]}}(z)$ to $G[B]$.

We have shown that (G_A, G_B) is non-trivial bipartite super-separation of G so that G_A and G_B both have at least five vertices, and neither G_A nor G_B is isomorphic to $K_{3,t}$ for any t. Furthermore, the order of (G_A, G_B) is $|S|$ and

$$\begin{aligned}|E(G_A)| + |E(G_B)| - |E(G)| &= |E(G[A])| + |E(G[B])| + 2d_{\overline{G[S]}}(z) - |E(G)| \\ &= |E(G[S])| + 2d_{\overline{G[S]}}(z)\end{aligned}$$

So by Lemma 3.4 applied to the super-separation (G_A, G_B), we have $3|S| - |E(G[S])| - 2d_{\overline{G[S]}}(z) \geq 11$. If $|S| \leq 3$, then $9 \geq 3|S| \geq 3|S| - |E(G[S])| - 2d_{\overline{G[S]}}(z) \geq 11$, a contradiction. Thus either $4 \leq |S| \leq 5$, or $|S| = 6$ and $\Delta(G[S]) \geq 2$.

Without loss of generality assume that $|X_0 \cap S| \leq |X_1 \cap S|$. Let $z' \in X_0 \cap S$ with $d_{\overline{G[S]}}(z')$ minimum. Note that if $|S| = 6$ we may assume that $d_{\overline{G[S]}}(z') = 3 - \Delta(G[S])$. We have:

$$\begin{aligned}3(|S|-3) &\leq |X_0 \cap S||X_1 \cap S| = \sum_{x \in X_0 \cap S} |X_1 \cap S| \\ &= \sum_{x \in X_0 \cap S} \big(d_{G[S]}(x) + |X_1 \cap S| - d_{G[S]}(x)\big) = \sum_{x \in X_0 \cap S} \Big(d_{G[S]}(x) + d_{\overline{G[S]}}(x)\Big) \\ &\leq \left(\sum_{x \in X_0 \cap S} d_{G[S]}(x)\right) + (|X_0 \cap S| - 1)d_{\overline{G[S]}}(z) + d_{\overline{G[S]}}(z') \\ &= |E(G[S])| + (|X_0 \cap S| - 1)d_{\overline{G[S]}}(z) + d_{\overline{G[S]}}(z')\end{aligned}$$

If $4 \leq |S| \leq 5$, then $|X_0 \cap S| \leq 2$ and so from above, we have $3|S| - 9 \leq |E(G[S])| + 2d_{\overline{G[S]}}(z)$. This is a contradiction since $3|S| - |E(G[S])| - 2d_{\overline{G[S]}}(z) \geq 11$. If $|S| = 6$ and $\Delta(G[S]) \geq 2$, then $d_{\overline{G[S]}}(z') \leq 1$ and so $3|S| - 9 \leq |E(G[S])| + 2d_{\overline{G[S]}}(z) + 1$. This is again a contradiction since $3|S| - |E(G[S])| - 2d_{\overline{G[S]}}(z) \geq 11$.

□

Next we observe that G has no $K_{3,3}$ subgraph, and then we show that common neighbors of a vertex of degree four or five in fact share several common neighbors.

Lemma 3.7. *G does not have a subgraph isomorphic to $K_{3,3}$.*

Proof. Suppose H is a subgraph of G isomorphic to $K_{3,3}$. Since $n \geq 7$ by Lemma 3.2, the graph $G - V(H)$ is non-empty. Let C be the vertex set of some component of $G - V(H)$. By Lemma 3.6, the separation $(C \cup N(C), V(G) - C)$ is trivial. Then $C \cup N(C) = V(G)$, and so $N(C) = V(H)$. So the graph obtained by contacting C to a single vertex is isomorphic to $K_{1,3,3}$, a contradiction. □

Lemma 3.8. *Let $v \in V(G)$ be a vertex of degree four or five. Let x and y be distinct vertices in $N(v)$. Then x and y share at least $7 - d(v)$ common neighbors other than v.*

Proof. Suppose otherwise, and write $c := |N(x) \cap N(y)| - 1$. That is, c is the number of common neighbors of x and y other than v. So we have $c \leq 6 - d(v)$. Without loss of generality suppose that $v \in X_0$. Let G' be the graph formed from G by deleting y and v, and adding edges between x and all vertices in $N(y) - N(x)$. We can see that G' is a minor of G by contracting y and v to x. Furthermore, G' is bipartite and since $n \geq 7$ by Lemma 3.2, the graph G' has at least five vertices. Let l be 1 if G' is isomorphic to $K_{3,t}$ for some t, and 0 otherwise. Then we have:

$$
\begin{aligned}
3n - 10 < |E(G)| &= |E(G - v)| + d(v) = |E(G')| + c + d(v) \\
&\leq 3(n-2) - 10 + l + c + d(v) = 3n - 10 + l + (c - 6 + d(v))
\end{aligned}
$$

It follows that $l = 1$ and $c = 6 - d(v)$. If $d(v) = 4$, then $c = 2$ and $G[N(v) \cup (N(x) \cap N(y))]$ is isomorphic to $K_{3,4}$. This is a contradiction since by Lemma 3.7, the graph G has no $K_{3,3}$ subgraph. If $d(v) = 5$, then $|X_1 \cap V(G')| \geq d(v) - 1 = 4$, and so $|X_0 \cap V(G')| = 3$. Then $G[(X_0 \cap V(G')) \cup (N(v) - \{x, y\})]$ is isomorphic to $K_{3,3}$, again a contradiction to Lemma 3.7. □

Now we are ready to show:

Lemma 3.9. *G has no vertex of degree five.*

Proof. Suppose $v \in V(G)$ is a vertex of degree five. Let

$$W \coloneqq \{w \in V(G) - N[v] : |N(w) \cap N(v)| = 2\}$$

and let

$$U_0 \coloneqq \{u \in V(G) - N[v] : |N(u) \cap N(v)| > 2\}.$$

Let G_0 be the graph formed from $G[N[v] \cup W \cup U_0]$ by contracting, for every vertex $w \in W$, an arbitrary edge with one end w and the other end in $N(w) \cap N(v)$. By Lemma 3.8, every pair of vertices in $N(v)$ are either adjacent in G_0 or share at least two common neighbors in U_0.

First we show the following claim:

Claim 3.9.1. *There exist a set $U_1 \subseteq U_0$ and a graph G_1 so that:*

(i) The graph G_1 is formed from G_0 by contracting edges with one end in $U_0 - U_1$ and the other end in $N(v)$.

(ii) Every pair of distinct vertices in $N(v)$ are either adjacent in G_1 or share a common neighbor in U_1.

(iii) Every vertex in U_1 has degree exactly three in G_0, and $\delta(G_1[N(v)]) \geq 1$.

Proof. Observe that U_0 is non-empty since otherwise G_0 is isomorphic to K_6. Fix a vertex $z \in U_0$ with $d_{G_0}(z)$ maximum. First suppose $d_{G_0}(z) = 5$. Then since G has no $K_{3,3}$ subgraph by Lemma 3.7, every pair of vertices in $N(v)$ are adjacent in G_0. Then $G_0[N[v]]$ is isomorphic to K_6, a contradiction. So $d_{G_0}(z) \leq 4$.

Now observe that every vertex in U_0 other than z has degree exactly three in G_0. This is clear if $d_{G_0}(z) = 3$, and follows since G has no $K_{3,3}$ subgraph if $d_{G_0}(z) = 4$.

Let $x \in N(v) - N_{G_0}(z)$. If $d_{G_0}(z) = 3$, let x' be the vertex other than x in $N(v) - N_{G_0}(z)$. If $d_{G_0}(z) = 4$, let x' be any vertex in $N(v)$ other than x.

First suppose that x and x' are adjacent in G_0. Then let G_1 be the graph formed from G_0 by contracting z to one of its neighbors in G_0, and let $U_1 \coloneqq U_0 - \{z\}$. Then G_1 and U_1 satisfy the conditions of the claim.

So we may assume that x and x' are not adjacent in G_0. Then they have a common neighbor $z' \in U_0 - \{z\}$. Let G_1 be the graph formed from G_0 by contracting z to a vertex in $N_{G_0}(z) - N_{G_0}(z')$ and z' to x'. Write $U_1 \coloneqq U_0 - \{z, z'\}$. Then G_1 and U_1 satisfy the conditions of the claim. □

Fix G_1 and U_1 as in the claim. Choose a graph G_2 and a set $U_2 \subseteq U_1$ so that:

(i) The graph G_2 is formed from G_1 by contracting edges with one end in $U_0 - U_1$ and the other end in $N(v)$.

(ii) Every pair of distinct vertices in $N(v)$ are either adjacent in G_2 or share a common neighbor in U_2.

(iii) Subject to the above, $|U_2|$ is minimum.

Such a choice is possible because $G_2 \coloneqq G_1$ and $U_2 \coloneqq U_1$ satisfy (i) and (ii). Observe that G_2 is a minor of G. We first show that for all $u \in U_2$, the graph $G_2[N_{G_2}(u)]$ has no edges. Since every vertex in U_1 has degree exactly three in G_0 by the claim, the vertex u also has degree exactly three in G_2. Write $N_{G_2}(u) = \{x, x', x''\}$ and suppose $xx' \in E(G_2)$. Then let G_2' be the graph formed from G_2 by contracting u to x'', and let $U_2' \coloneqq U_2 - \{u\}$. Then G_2' and U_2' show a contradiction to our choice of G_2 and U_2.

Then by the last paragraph and condition (ii), if $|U_2| \leq 1$, then G_2 is isomorphic to either K_6 or $K_6^{\Delta Y}$. So there exist distinct vertices $u, u' \in U_2$. Both u and u' have degree exactly three in G_2. We go by cases.

Case: $|N_{G_2}(u) \cap N_{G_2}(u')| = 3$

Then $G[N_{G_2}(u) \cup \{v, u, u'\}]$ is isomorphic to $K_{3,3}$, a contradiction to Lemma 3.7.

Case: $|N_{G_2}(u) \cap N_{G_2}(u')| = 2$

Then let x be the unique vertex in $N_{G_2}(u) - N_{G_2}(u')$. Let G_2' be the graph formed from G_2 by contracting u to x, and let $U_2' \coloneqq U_2 - \{u\}$. Then G_2' and U_2' show a contradiction to our choice of G_2 and U_2.

Case: $|N_{G_2}(u) \cap N_{G_2}(u')| = 1$

Let x be the unique vertex in $N_{G_2}(u) \cap N_{G_2}(u')$. Then x is adjacent to no vertices in $N(v)$ in the graph G_2. But this is a contradiction since $\delta(G_2[N(v)]) \geq \delta(G_1[N(v)]) \geq 1$ by part (iii) of Claim 3.9.1. This is the final case and completes the proof of the lemma. □

4 Proof of Main Theorem: Vertices of Degree Four

Now that we have shown G has no vertices of degree five and that $4 \leq \delta(G) \leq 5$ by Lemma 3.3, the remainder of the proof deals with vertices of degree four. First we will show that if v is any vertex of degree four, then G has no vertex u such that $|N(u) \cap N(v)| \geq 4$. We then use this fact to show that G does not have additional kinds of separations of small order. Finally we fix a vertex v of degree four and a certain set $U \subseteq V(G) - N[v]$ of three or fewer vertices that each have neighbors in $N(v)$. We show that $G - (N[v] \cup U)$ is connected and has a cut vertex a. We then use the fact that $G - (N[v] \cup U \cup \{a\})$ is disconnected to find a separation showing a contradiction to Lemma 3.4.

Lemma 4.1. *Suppose $v \in V(G)$ is a vertex of degree four. Then there does not exist a vertex $u \in V(G) - N[v]$ so that $|N(u) \cap N(v)| = 4$.*

Proof. Suppose otherwise. Without loss of generality assume that $v \in X_0$. Write $N(v) = \{v_1, v_2, v_3, v_4\}$. For every $i, j \in \{1, 2, 3\}$ with $i < j$, let $u_{i,j} \in V(G) - \{v, u\}$ be a vertex that is adjacent to both v_i and v_j. Such vertices exist since by Lemma 3.8, v_i and v_j have at least three common neighbors other than v. Since G has no $K_{3,3}$ subgraph by Lemma 3.7 and $|N(u) \cap N(v)| = 4$, the vertices $u_{1,2}$, $u_{1,3}$, and $u_{2,3}$ are distinct. Write $U := \{u_{1,2}, u_{1,3}, u_{2,3}\}$, and $H := G[N[v] \cup U \cup \{u\}]$. Then $d_H(v_4) < 3$ since G has no $K_{3,3}$ subgraph. So since $\delta(G) \geq 4$ by Lemma 3.3, there exists a component of $G - V(H)$ with neighbor v_4. Let C be the vertex set of such a component. Observe that $N(C) \subseteq N(v) \cup U \cup \{u\}$.

Now we show that either $N(v) \subseteq N(C)$ or $U \cup \{u\} \subseteq N(C)$. Suppose otherwise. Then for all $i \in \{0, 1\}$, we have $|X_i \cap N(C)| \leq 3$. Then by Lemma 3.6 applied to the separation $(C \cup N(C), V(G) - C)$, it follows that $|N(C)| = 6$ and $\Delta(G[N(C)]) \leq 1$. Then since $|N(C) \cap N(v)| = 3$ and $|N(u) \cap N(v)| = 4$, we have $u \notin N(C)$. Then $U \subseteq N(C)$. But $|N(C) \cap \{v_1, v_2, v_3\}| \geq 2$, which is a contradiction since $\Delta(G[N(C)]) \leq 1$. We have shown that either $N(v) \subseteq N(C)$ or $U \cup \{u\} \subseteq N(C)$.

If $N(v) \subseteq N(C)$, let G' be the graph formed from G by contracting C to a single vertex with neighborhood $N(v)$ and deleting all other vertices in $G - V(H)$. Let G'' be the graph formed from G' by contracting $u_{i,j}$ to v_i for all $i, j \in \{1, 2, 3\}$ with $i < j$. Then G'' is isomorphic to $K_6^{\Delta Y}$, a contradiction.

So we may assume that $U \cup \{u\} \subseteq N(C)$. Remember also that by the choice of C, we have $v_4 \in N(C)$. Now let G' be the graph formed by contracting C to a vertex with neighborhood $U \cup \{u, v_4\}$ and deleting all other vertices in $G - V(H)$. Let G'' be the graph formed from G' by contracting v_4 to v and by contracting $u_{i,j}$ to v_i for all $i, j \in \{1, 2, 3\}$ with $i < j$. Then G'' is isomorphic to K_6, a contradiction. □

We are now ready to show that G does not have additional kinds of separations of small order.

Lemma 4.2. *Let (A, B) be a non-trivial separation of G. If there exists $i \in \{0, 1\}$ such that $|X_i \cap A \cap B| \leq 4$ and $|X_{1-i} \cap A \cap B| \leq 2$, then either $|A - B| = 1$ or $|B - A| = 1$.*

Proof. Suppose otherwise. Let (A, B) be a separation of minimum order that violates the lemma. Write $S := A \cap B$ for convenience. Without loss of generality we assume that $|X_0 \cap S| \leq 4$ and $|X_1 \cap S| \leq 2$. By Lemma 3.6, we have $|X_0 \cap S| = 4$.

First we will show that there exists a component of $G[A - B]$ with neighborhood S. Suppose otherwise. Let C be the vertex set of any component of $G[A - B]$. If $|C| \geq 2$ and $N(C) \neq S$, then $(C \cup N(C), V(G) - C)$ is a separation violating the lemma of smaller order, a contradiction to our choice of (A, B). So $|C| = 1$ and thus since $\delta(G) \geq 4$ by Lemma 3.3, we have $N(C) = X_0 \cap S$. Then since $|A - B| \geq 2$, the graph $G[A - B]$ has another component with vertex set C' also consisting of a single vertex with neighborhood $X_0 \cap S$. But this is a contradiction to Lemma 4.1. This shows that there exists a component of $G[A - B]$ with neighborhood S. By symmetry the same holds for $G[B - A]$. We now proceed by cases.

Case: Either $|S| = 4$, or $|S| = 5$ and $|E(G[S])| = 4$

Let G_A be the graph formed from $G[A]$ by adding a single vertex, call it a, with neighborhood $X_0 \cap S$. We can see that G_A is a minor of G by contracting a component of $G[B - A]$ with neighborhood S to a single vertex. Furthermore, G_A is bipartite, has fewer vertices than G since by assumption $|B - A| > 1$, and has at least five vertices since $A \subseteq V(G_A)$. Define G_B analogously, by adding a single vertex with neighborhood $S \cap X_0$ to $G[B]$.

Suppose G_A is isomorphic to $K_{3,t}$ for some t. Then since $N_{G_A}(a) = S \cap X_0$ and $|S \cap X_0| = 4$, there exist two vertices u and v in $G_A - N_{G_A}[a]$ with degree exactly four in G_A. Then since $|S| \leq 5$, we have $|S - N_{G_A}[a]| \leq 1$, so at least one of the vertices, say v, is in $A - S$. Then v has degree four in G, and $|N(u) \cap N(v)| = 4$, a contradiction to Lemma 4.1. By symmetry, we have shown that neither G_A nor G_B is isomorphic to $K_{3,t}$ for any t. Furthermore, the order of the super-separation (G_A, G_B) is $|S| + 2$, and

$$\begin{aligned}|E(G_A)| + |E(G_B)| - |E(G)| &= |E(G[A])| + |E(G[B])| + 8 - |E(G)| \\ &= |E(G[S])| + 8\end{aligned}$$

So by Lemma 3.4 applied to the super-separation (G_A, G_B), we have $3(|S| + 2) - |E(G[S])| - 8 \geq 11$. So $13 \leq 3|S| - |E(G[S])|$. But this is a contradiction since either $|S| \leq 4$, or $|S| = 5$ and $|E(G[S])| = 4$.

Case: Either $|S| = 5$ and $|E(G[S])| < 4$, or $|S| = 6$

This case is similar to the proof of Lemma 3.6. Let $z \in X_1 \cap S$ so that $d_{\overline{G[S]}}(z)$ is maximum. Let G_A be the graph formed from $G[A]$ by adding edges between z and every vertex in $N_{\overline{G[S]}}(z)$. We can see that G_A is a minor of G by contracting a component of $G[B - A]$ with neighborhood S to the vertex z. The graph G_A is bipartite, has fewer vertices than G, and has at least four vertices in each side of the bipartition of G by Lemma 3.5 applied to the separation (A, B) since $V(G_A) = A$. Define G_B analogously, by adding edges between z and every vertex in $N_{\overline{G[S]}}(z)$ to $G[B]$.

The order of the super-separation (G_A, G_B) is $|S|$, and

$$\begin{aligned}|E(G_A)| + |E(G_B)| - |E(G)| &= |E(G[A])| + |E(G[B])| + 2d_{\overline{G[S]}}(z) - |E(G)| \\ &= |E(G[S])| + 2d_{\overline{G[S]}}(z)\end{aligned}$$

Then by Lemma 3.4, we have $3|S| - |E(G[S])| - 2d_{\overline{G[S]}}(z) \geq 11$. Observe that $4|X_1 \cap S| = \sum_{x \in X_1 \cap S} \left(d_{G[S]}(x) + d_{\overline{G[S]}}(x) \right)$. So if $|S| = 5$ and $|E(G[S])| \leq 3$, then $d_{\overline{G[S]}}(z) \geq 1$ and so $|E(G[S])| + 2d_{\overline{G[S]}}(z) \geq 5$. This is a contradiction. So $|S| = 6$. But then $|E(G[S])| + 2d_{\overline{G[S]}}(z) \geq 8$, which is again a contradiction. $\square$

By Lemmas 3.3 and 3.9, the graph G has a vertex of degree four. Fix $v \in V(G)$ a vertex of degree four, and write $N(v) = \{v_1, v_2, v_3, v_4\}$. Without loss of generality assume that $v \in X_0$. Choose a set $U \subseteq V(G) - N[v]$ of minimum cardinality such that either:

(i) U consists of a single vertex u with $|N(u) \cap N(v)| = 3$, or

(ii) $U = \{u_{1,2}, u_{1,3}, u_{2,3}\}$ and for all $i, j \in \{1, 2, 3\}$ with $i < j$, $N(u_{i,j}) \cap N(v) = \{v_i, v_j\}$.

First we show that such a set exists. If there exists a vertex $u \in V(G) - N[v]$ such that $|N(u) \cap N(v)| \geq 3$, then by Lemma 4.1 in fact $|N(u) \cap N(v)| = 3$ and we are done. So we may assume that for all $u \in V(G) - N[v]$ we have $|N(u) \cap N(v)| < 3$. Then for all $i, j \in \{1, 2, 3\}$ with $i < j$, let $u_{i,j}$ be a vertex not in $N[v]$ that is adjacent to both v_i and v_j. Such a vertex exists since v_i and v_j have at least three common neighbors other than v by Lemma 3.8. By assumption $u_{1,2}$, $u_{1,3}$, and $u_{2,3}$ are distinct and $N(u_{i,j}) \cap N(v) = \{v_i, v_j\}$. So such a set exists.

Write $H \coloneqq G[N[v] \cup U]$. Next we show one short lemma.

Lemma 4.3. *There do not exist disjoint sets $A, B \subseteq V(G) - V(H)$ such that $G[A]$ and $G[B]$ are connected, and $N(v) \subseteq N(A)$ and $N(v) \subseteq N(B)$.*

Proof. Let G' be the graph obtained from G by contracting A to a single vertex with neighborhood $N(v)$, contracting B to a single vertex with neighborhood $N(v)$, and deleting all other vertices in $G - V(H)$.

If $|U| = 1$, then G' is isomorphic to $K_{4,4}^-$, a contradiction. If $|U| = 3$, then let G'' be the graph formed from G' by contracting $u_{i,j}$ to v_i for all $i, j \in \{1, 2, 3\}$ with $i < j$. Then G'' is isomorphic to $K_6^{\Delta Y}$, a contradiction. ☐

In the next lemma we show that $G - V(H)$ is connected and has a 1-separation satisfying certain properties.

Lemma 4.4. *The graph $G - V(H)$ is connected. Furthermore, there exist $\{a_0, a_0', a_1, a_1'\} \subseteq V(G) - V(H)$ and a 1-separation (A_0, A_1) of $G - V(H)$ such that for every $i \in \{0, 1\}$, we have $a_i, a_i' \in A_i$ and a_i and a_i' are both adjacent to v_{2i+1} and v_{2i+2}.*

Proof. First we will show that $G - V(H)$ is connected. Otherwise, by Lemma 4.3, there exists a component of $G - V(H)$ with vertex set C so that $N(v) \nsubseteq N(C)$. So by Lemma 3.6 applied to the separation $(C \cup N(C), V(G) - C)$, we find that $|N(C)| = 6$ and $\Delta(G[N(C)]) \leq 1$. It follows that $|U| = 3$ and $U \subseteq N(C)$. This is a contradiction since $|N(C) \cap \{v_1, v_2, v_3\}| \geq 2$ and $\Delta(G[N(C)]) \leq 1$. So $G - V(H)$ is connected.

Now for every $i \in \{0, 1\}$, let a_i and a_i' be distinct vertices in $V(G) - V(H)$ that are adjacent to both v_{2i+1} and v_{2i+2}. Such vertices exist since by Lemma 3.8 the vertices v_{2i+1} and v_{2i+2} share at least three common neighbors other than v, and by the definition of U they share no more than one common neighbor in U. Furthermore by Lemma 4.1, in fact a_0, a_0', a_1, a_1' are all distinct. By Menger's Theorem, either the desired 1-separation exists, or $G - V(H)$ contains vertex-disjoint paths P and P' so that both P and P' have one end in $\{a_0, a_0'\}$ and one end in $\{a_1, a_1'\}$. But then by choosing $A := V(P)$ and $B := V(P')$ we have a contradiction to Lemma 4.3. □

Fix $\{a_0, a_0', a_1, a_1'\} \subseteq V(G) - V(H)$ and a 1-separation (A_0, A_1) of $G - V(H)$ as in the lemma. Let a be the unique vertex in $A_0 \cap A_1$, and for convenience write $H' := G[V(H) \cup \{a\}]$. Let C be the vertex set of a component of $G - V(H')$ so that $1 \leq |N(C) \cap N(v)| < 4$. Subject to this, choose C such that $|N(C)|$ is minimum.

To see that such a component exists, for every $i \in \{0, 1\}$, let C_i be the vertex set of a component of $G[A_i - \{a\}]$ with $C_i \cap \{a_i, a_i'\} \neq \emptyset$. Then $G[C_0]$ and $G[C_1]$ are distinct components of $G - V(H')$. By Lemma 4.3, either $N(v) \nsubseteq N(C_0)$ or $N(v) \nsubseteq N(C_1)$. So such a component exists. We first show:

Lemma 4.5. $|U| = 3$

Proof. Suppose $|U| = 1$. Let u be the unique vertex in U. Without loss of generality we may assume that $N(u) \cap N(v) = \{v_1, v_2, v_3\}$. Remember that $v \in X_0$. If $a \in X_0$, then for every $i \in \{0, 1\}$, we have $|N(C) \cap X_i| \leq 3$. But we have $|N(C)| \leq 5$, which is a contradiction to Lemma 3.6. Thus we have $a \in X_1$. We prove the following claim:

Claim 4.5.1. *Let C' be the vertex set of a component of $G - V(H')$ so that $N(v) \nsubseteq N(C')$. Then C' consists of a single vertex of degree four that is only adjacent to vertices in $N(v) \cup \{a\}$.*

Proof. Let C' be the vertex set of such a component. Then $|N(C') \cap X_1| = |N(C') \cap (N(v) \cup \{a\})| \leq 4$ and $|N(C') \cap X_0| = |N(C') \cap U| \leq 1$. Note that $|V(G) - V(C')| \geq |N[v] - V(C')| \geq 2$. Then by Lemma 4.2 applied to the separation $(C' \cup N(C'), V(G) - C')$, the set C' consists of a single vertex. Then since $\delta(G) \geq 4$ by Lemma 3.3, the vertex in C' is only adjacent to vertices in $N(v) \cup \{a\}$. □

Now define the set

$$W := \{w \in V(G) - V(H') : d_G(w) = 4 \text{ and } N(w) \subseteq N(v) \cup \{a\}\}.$$

Since $G - V(H)$ is connected by Lemma 4.1, every vertex $w \in W$ is adjacent to a.

Now we show that $|W| \geq 2$. By the claim and the choice of C, we have $|C| = 1$. Since $a \in X_1$ while $\{a_0, a_0', a_1, a_1'\} \subseteq X_0$, for every $i \in \{0, 1\}$ we have $|A_i - (\{a\} \cup C)| \geq |\{a_i, a_i'\} - C| \geq 1$. So $G - V(H')$ has at least three components. So by Lemma 4.3 and by the claim, we have $|W| \geq 2$.

Next we show that $G - (V(H') \cup W)$ has a component with vertex set D so that $N(v) \cup \{a\} \subseteq N(D)$. By Lemma 3.8, the vertices v_1 and v_2 have at least three common neighbors besides v. Since G has no $K_{3,3}$ subgraph by Lemma 3.7 and by the definition of W, the vertices v_1 and v_2 have a common neighbor in $V(G) - (V(H') \cup W)$. So $V(G) - (V(H') \cup W)$ is non-empty. Let D be the vertex set of any component of $G - (V(H') \cup W)$. Since $G - V(H)$ is connected, we have $a \in N(D)$. If $N(v) \not\subseteq N(D)$, then by the claim, D consists of a single vertex of degree four that is only adjacent to vertices in $N(v) \cup \{a\}$. But this is a contradiction to the choice of W. So $N(v) \cup \{a\} \subseteq N(D)$.

Let w and w' be distinct vertices in W. Since G has no $K_{3,3}$ subgraph, we may assume without loss of generality that $N(w) = \{v_1, v_2, v_4, a\}$ and $N(w') = \{v_1, v_3, v_4, a\}$. Then let G' be the graph formed from G by contracting D to a single component with neighborhood $N(v) \cup \{a\}$ and deleting all other vertices except $V(H') \cup \{w, w'\}$. Then let G'' be the graph formed from G' by contracting w to v_2, u to v_3, v to v_4, and w' to a. Then G'' is isomorphic to K_6, a contradiction. This completes the proof of the lemma. □

So $|U| = 3$. Write $U = \{u_{1,2}, u_{1,3}, u_{2,3}\}$ so that for all $i, j \in \{1, 2, 3\}$ with $i < j$, $N(u_{i,j}) \cap N(v) = \{v_i, v_j\}$. By the choice of U, no vertex other than v is adjacent to three or more vertices in $N(v)$. For convenience write $T \coloneqq N_{G[N(C)]}(a) \cup N_{\overline{G[N(C)]}}(a)$. That is, T is the set of all vertices in $N(C)$ that are in the other side of the bipartition of G as the vertex a. Let x be some vertex in $N(v) - N(C)$. Such a vertex exists since by the choice of C we have $|N(C) \cap N(v)| < 4$.

Now we give an overview of the rest of the proof. The goal is to show a contradiction to Lemma 3.4 on super-separations of G. Note that since $|N(C) \cap N(v)| < 4$, we have $N(C) \subsetneq N(v) \cup U \cup \{a\}$. So $|N(C)| \leq 7$. The previous lemmas on separations of G, Lemmas 3.6 and 3.6, apply only to separations of order six or less, so some casework is required to show a contradiction. We will frequently construct a super-separation (G_C, G') so that $G[C \cup N(C)]$ is a subgraph of G_C and $G - C$ is a subgraph of G'.

We first show a straightforward lemma that will help with constructing such super-separations. We are then able to show the harder lemma that $a \in X_1$ and that $G - (V(H' \cup C))$ is connected. Then it is easy to show that $v_4 \notin N(C)$, or else G has a K_6 minor. A final lemma shows that certain vertices in U have no neighbor in $V(G) - (V(H') \cup C)$. We then construct one last super-separation of G that gives a contradiction to Lemma 3.4, completing the proof. We begin with the following lemma.

Lemma 4.6. *The following hold:*

(i) The set C has at least two vertices. Both the set $C \cup N(C)$ and the set $V(G) - C$ have at least four vertices in each side of the bipartition of G.

(ii) Every neighbor of v is adjacent to a vertex in $V(G) - (V(H') \cup C)$.

(iii) $|N(C) \cap N(v)| = 3$ and $|T| = 3$

Proof. First we show that $|C| > 1$ and $|V(G) - (V(H') \cup C)| > 1$. We have $|V(G) - (C \cup N(C))| \geq |N[v] - (C \cup N(C))| \geq 2$. Suppose $|C| = 1$. Since $|N(C) \cap N(v)| \geq 1$ by the choice of C, it follows that $N(C) \subseteq N(v) \cup \{a\}$. But then since $\delta(G) \geq 4$ by Lemma 3.3, we have $|N(C) \cap N(v)| \geq 3$, a contradiction to the choice of U.

Next we show (i). The set $V(G) - C$ has at least four vertices in each side of the bipartition of G since $V(H) \subset V(G) - C$. Since $|C| > 1$ and $G[C]$ is connected, the set C is not contained in one side of the bipartition of G. Thus $C \cup N(C)$ has at least four vertices in each side of the bipartition of G by Lemma 3.5.

Now we show (ii). Let y be any vertex in $N(v) - \{x\}$. By Lemma 3.8, the vertices x and y share at least three common neighbors other than v. By the choice of U, they share no more than two common neighbors in $U \cup \{a\}$. So x and y have a common neighbor in $V(G) - (V(H') \cup C)$.

Finally we show (iii). If $a \in X_0$ and $|N(C) \cap N(v)| < 3$, this is a contradiction to Lemma 4.2 applied to the separation $(C \cup N(C), V(G) - C)$. If $a \in X_1$ and $|N(C) \cap N(v)| < 3$, then by Lemma 3.6, we have $|N(C)| = 6$ and $\Delta(G[N(C)]) \leq 1$. This is a contradiction since then $U \subseteq N(C)$ and $|N(C) \cap N(v)| = 2$, but every vertex in $\{v_1, v_2, v_3\}$ has two neighbors in U.

If $a \in X_0$, then $T = N(C) \cap N(v)$ and so $|T| = 3$ by the last paragraph. If $a \in X_1$ and $|T| < 3$, then this is a contradiction to Lemma 4.2 applied to the separation $(C \cup N(C), V(G) - C)$. □

Next we show the following lemma.

Lemma 4.7. *$a \in X_1$. Furthermore, $G - (V(H') \cup C)$ is connected.*

Proof. Suppose otherwise. That is, suppose that either $a \in X_0$ or $G - (V(H') \cup C)$ is not connected. Then:

Claim 4.7.1. *If $a \in X_1$, then there exists a component of $G - (V(H') \cup C)$ with vertex set C' so that $U \subseteq N(C')$ and $|N(C') \cap N(v)| = 3$.*

Proof. Suppose $a \in X_1$. Then $G - (V(H') \cup C)$ is not connected, and so by Lemma 4.3, it has a component with vertex set C' such that $N(v) \nsubseteq N(C')$. Then by the choice of C, we have $|N(C')| \geq |N(C)|$. By part (iii) of Lemma 4.6, since $a \in X_1$, we have $|N(C)| = 7$. Then $|N(C')| \geq 7$, and so $U \subseteq N(C')$ and $|N(C') \cap N(v)| = 3$. □

Recall that by (iii) of Lemma 4.6, we have $|N(C) \cap N(v)| = 3$. If $v_4 \notin N(C)$, then let G_C denote the graph obtained from $G[C \cup N(C)]$ by adding edges between a and every vertex in $N_{\overline{G[N(C)]}}(a)$. If $v_4 \in N(C)$, then let G_C denote the graph obtained from $G[C \cup N(C)]$ by adding edges between a and every vertex in $N_{\overline{G[N(C)]}}(a)$, and by adding edges between v_4 and every vertex in $N(x) \cap U \cap N(C)$.

Observe that in either case the graph G_C is bipartite and has fewer vertices than G. Furthermore, by part (i) of Lemma 4.6, the graph G_C is not isomorphic to $K_{3,t}$ for any t, and has at least five vertices. We now show two claims about the graph G_C.

Claim 4.7.2. *The graph G_C is a minor of G.*

Proof. Recall that $G - V(H)$ is connected by Lemma 4.4. So every component of $G - (V(H') \cup C)$ has the vertex a as a neighbor.

First suppose that $a \in X_0$. By part (ii) of Lemma 4.6, every vertex in $N(v)$ has a neighbor in $V(G) - (V(H') \cup C)$. Then we can see that G_C is a minor of G by contracting every component of $G - (V(H') \cup C)$ to the vertex a, and if $v_4 \in N(C)$, by contracting x and v to v_4.

So we may assume that $a \in X_1$. Then by Claim 4.7.1, there exists a component of $G - (V(H') \cup C)$ with vertex set C' so that $U \subseteq N(C')$ and $|N(C') \cap N(v)| = 3$. Then we can see that G_C is a minor of G by contracting C' to the vertex a, and if $v_4 \in N(C)$, by contracting x and v to v_4. □

Next we prove that the following inequality holds.

Claim 4.7.3. $|E(G_C)| + |E(G - C)| - |E(G)| \geq 3 + 2|N(C) \cap U|$

Proof. First suppose that $|N(C) \cap U| = 3$. Observe that then if $v_4 \in N(C)$ we have $|N(x) \cap U \cap N(C)| = |N(x) \cap U| = 2$. Let $\mathbb{1}_{v_4 \in N(C)}$ be one if $v_4 \in N(C)$ and zero otherwise. Then we have:

$$\begin{aligned}
&|E(G_C)| + |E(G - C)| - |E(G)| \\
&\quad = d_{\overline{G[N(C)]}}(a) + 2\mathbb{1}_{v_4 \in N(C)} + |E(G[C \cup N(C)])| + |E(G - C)| - |E(G)| \\
&\quad = d_{\overline{G[N(C)]}}(a) + 2\mathbb{1}_{v_4 \in N(C)} + |E(G[N(C)])| \\
&\quad = 2\mathbb{1}_{v_4 \in N(C)} + |T| + |E(G[N(C) - \{a\}])|
\end{aligned}$$

In either case, since $|T| = 3$ by part (iii) of Lemma 4.6, we have

$$2\mathbb{1}_{v_4 \in N(C)} + |T| + |E(G[N(C) - \{a\}])| = 9 = 3 + 2|N(C) \cap U|$$

which completes the case that $|N(C) \cap U| = 3$.

So we may assume that $|N(C) \cap U| \leq 2$. Then since $|T| \geq 3$, it follows that $a \in X_0$. Then by Lemma 3.6 applied to the separation $(C \cup N(C), V(G) - C)$, we have $|N(C) \cap U| = 2$ and $\Delta(G[N(C)]) \leq 1$. By symmetry between pairs of vertices in U, we may assume that $N(C) \cap U = \{u_{1,2}, u_{1,3}\}$. Then $v_1 \notin N(C)$. Then $|N(x) \cap U \cap N(C)| = |N(v_1) \cap U \cap N(C)| = 2$ and similarly to the last case we find that

$$\begin{aligned}
|E(G_C)| + |E(G - C)| - |E(G)| &= |T| + 2 + |E(G[N(C) - \{a\}])| \\
&= |T| + 4 = 3 + 2|N(C) \cap U|
\end{aligned}$$

which completes the proof of the claim. □

The final claim we show is:

Claim 4.7.4. $\Delta(\overline{G[N(C)]}) \leq 1$

Proof. Suppose that $\Delta(\overline{G[N(C)]}) \geq 2$. Let $z \in N(C)$ be a vertex with maximum degree in $\overline{G[N(C)]}$. Then let G' be the graph obtained from $G - C$ by adding an edge between z and every vertex in $N_{\overline{G[N(C)]}}(z)$. We can see that G' is a minor of G on strictly fewer vertices by contracting C to the vertex z. Furthermore, G' is bipartite, and by part (i) of Lemma 4.6, has at least five vertices and is not isomorphic to $K_{3,t}$ for any t. By Claim 4.7.2, the graph G_C is a minor of G. So since $(C \cup N(C), V(G) - C)$ is a separation of G, it follows that (G_C, G') is a super-separation of G.

In fact we have shown that (G_C, G') is a non-trivial bipartite super-separation of G so that both G_C and G' have at least five vertices and are not isomorphic to $K_{3,t}$ for any t. Since $|N(C) \cap N(v)| = 3$ and $a \in N(C)$, the order of the super-separation (G_C, G') is $4 + |N(C) \cap U|$. Then by Lemma 3.4 and the inequality from the last claim, we have

$$\begin{aligned} 1 + 3|N(C) \cap U| = 3(4 + |N(C) \cap U|) - 11 &\geq |E(G_C)| + |E(G')| - |E(G)| \\ &\geq |E(G_C)| + 2 + |E(G - C)| - |E(G)| \\ &\geq 5 + 2|N(C) \cap U| \end{aligned}$$

But then $|N(C) \cap U| \geq 4$, which is a contradiction since $|U| = 3$. □

We are now ready to complete the proof of the lemma. We go by cases.

Case: $a \in X_0$

By the last claim, we have $d_{\overline{G[N(C)]}}(a) \leq 1$. Then since $|N(C) \cap N(v)| = 3$, the vertex a is adjacent to at least two vertices in $N(v)$. So by the choice of U, the vertex a is adjacent to exactly two vertices in $N(v)$. So there exists a vertex $y \in N(C) \cap N(v)$ that is not adjacent to a. By the last claim, $d_{\overline{G[N(C)]}}(y) \leq 1$. So since y is not adjacent to a, the vertex y is adjacent to every vertex in $N(C) \cap U$.

Then $U \nsubseteq N(C)$. Then by Lemma 3.6 applied to the separation $(C \cup N(C), V(G) - C)$, we have $|N(C) \cap U| = 2$ and $\Delta(G[N(C)]) \leq 1$. But then $1 \geq d_{G[N(C)]}(y) = 3 - d_{\overline{G[N(C)]}}(y) \geq 2$, a contradiction.

Case: $a \in X_1$

Then since $|T| = 3$, we have $U \subseteq N(C)$. Suppose there exists $u \in U$ such that $ua \notin E(G)$. Then since $|N(C) \cap N(v)| = 3$ and u is adjacent to exactly two vertices in $N(v)$, it follows that $d_{\overline{G[N(C)]}}(u) \geq 2$, a contradiction to the last claim. So $U \subseteq N(a)$. Also by the last claim applied to the vertex v_4, we have $v_4 \notin N(C)$. So $N(C) = U \cup \{v_1, v_2, v_3, a\}$.

By Claim 4.7.1, there exists a component of $G - (V(H') \cup C)$ with vertex set C' so that $U \subseteq N(C')$ and $|N(C') \cap N(v)| = 3$. By symmetry between the vertices v_1, v_2, and v_3, we may assume that $v_1 \in N(C')$. Then let G'_C be the graph formed from $G[C \cup N(C)]$ by adding an edge between v_1 and $u_{2,3}$. We can see that G'_C is a minor of G with strictly fewer vertices by contracting C' to the vertex v_1. By part (i) of Lemma 4.6, the graph G'_C has at least five vertices and is not isomorphic to $K_{3,t}$ for any t.

Let G' be the graph formed from $G - C$ by adding an edge between v_1 and $u_{2,3}$. We can see that G' is a minor of G on strictly fewer vertices by contacting C to the vertex v_1. By part (i) of Lemma 4.6, the graph G' has at least five vertices and is not isomorphic to $K_{3,t}$ for any t.

Then (G'_C, G') is a non-trivial bipartite super-separation of G such that neither G'_C nor G' is isomorphic to $K_{3,t}$ for any t. So by Lemma 3.4 and since $U \subseteq N(a)$ and $N(C) = U \cup \{v_1, v_2, v_3, a\}$, we have:

$$\begin{aligned} 10 = 3(7) - 11 &\geq |E(G'_C)| + |E(G')| - |E(G)| \\ &= 2 + |E(G[C \cup N(C)])| + |E(G - C)| - |E(G)| \\ &= 2 + |E(G[N(C)])| = 11 \end{aligned}$$

a contradiction. This completes the proof of the lemma. □

We have shown that $a \in X_1$ and that $G - (V(H') \cup C)$ is connected. For convenience write $D := V(G) - (V(H') \cup C)$. By part (ii) of Lemma 4.6, we have $N(v) \subseteq N(D)$. Also since $|T| = 3$, we have $U \subseteq N(C)$. The final two lemmas show that certain vertices are not neighbors of C or D.

Lemma 4.8. $v_4 \notin N(C)$

Proof. Suppose $v_4 \in N(C)$. Remember that $U \cup \{a\} \subseteq N(C)$. Then let G' be the graph formed from G by contracting D to a single vertex with neighborhood $N(v) \cup \{a\}$ and by contracting C to a single vertex, call it c, with neighborhood $U \cup \{v_4, a\}$. Then let G'' be the graph formed from G' by contracting $u_{1,2}$ to v_1, $u_{2,3}$ to v_2, $u_{1,3}$ to v_3, v to v_4, and finally c to a. Then G'' is isomorphic to K_6, a contradiction. □

Lemma 4.9. *If $u \in U \cap N(D)$, then $ua \in E(G)$.*

Proof. Suppose otherwise. Let G_C be the graph formed from $G[C \cup N(C)]$ by adding edges between u and all vertices in $N_{\overline{G[N(C)]}}(u)$. We can see that G_C is a minor of G by contracting D to the vertex u, since $N(v) \cup \{a\} \subseteq N(D)$. Let G' be the graph formed from $G - C$ by adding edges between a and all vertices in $N_{\overline{G[N(C)]}}(a)$. We can see that G' is a minor of G by contracting C to the vertex a. Then (G_C, G') is a non-trivial bipartite super-separation of G of order $|N(C)| = 7$. By part (i) of Lemma 4.6, both G_C and G' have at least four vertices on each side of the bipartition of G. So neither is isomorphic to $K_{3,t}$ for any t. So by Lemma 3.4 and since u is adjacent to exactly two vertices in $N(v)$, we have

$$\begin{aligned}10 = 3(7) - 11 &\geq |E(G_C)| + |E(G')| - |E(G)| \\ &= d_{\overline{G[N(C)]}}(u) + d_{\overline{G[N(C)]}}(a) + |E(G[N(C)])| \\ &\geq 2 + |N(C) \cap U| + |E(G[N(C) - \{a\}])|\end{aligned}$$

Since $v_4 \notin E(G)$ by Lemma 4.8, we have $|E(G[N(C) - \{a\}])| = 6$. But then $2 + |N(C) \cap U)| + |E(G[N(C) - \{a\}])| = 11$, a contradiction. □

Write $U' := U \cap N(D)$. Let G_C be the graph formed from $G[C \cup N(C)]$ by adding a vertex with neighborhood $\{v_1, v_2, v_3, a\}$. We can see that G_C is a minor of G on strictly fewer vertices by contracting D to a single vertex and since $|D| > 1$ by the choice of U. Also, by part (i) of Lemma 4.6, the graph G_C has at least four vertices in each side of the bipartition of G.

Let G' be the graph formed from $G[D \cup N(D) \cup \{v\}]$ by adding a vertex with neighborhood $\{v_1, v_2, v_3, a\}$. We can see that G' is a minor of G on strictly fewer vertices by contracting C to a single vertex and since $|C| > 1$ by part (i) of Lemma 4.6. Furthermore, G' is not isomorphic to $K_{3,t}$ for any t since $va \notin E(G')$.

Now we show that every edge of H' is an edge of either G_C or G'. Let e be an edge of H'. If e is incident to v, then e is an edge of G'. If e is incident to a vertex in U, then e is an edge of the graph G_C. Furthermore, if e is incident to a vertex in U', then e is also an edge of G'.

So (G_C, G') is a non-trivial bipartite super-separation of G. Furthermore, neither G_C nor G' is isomorphic to $K_{3,t}$ for any t, and the order of the super-separation (G_C, G') is $6 + |U'|$. Remember also that by Lemma 4.9, every vertex $u \in U'$ is adjacent to the vertex a. Then by Lemma 3.4 we have

$$7 + 3|U'| = 3(6 + |U'|) - 11 \geq |E(G_C)| + |E(G')| - |E(G)| = 8 + 3|U'|,$$

a contradiction. This completes the proof of the theorem. □

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