Addendum to "Directed Tree-Width"

by Johnson, Robertson, Seymour and Thomas

There are two places in need of correction. We are greatful to Jan Obdrzalek for bringing the first to our attention.

In the second paragraph of Section 4, after the first line, the following definition should be insered: Let $Z \subseteq V(G)$ be not necessarily disjoint from S. The set S is Z-*regular* if there is no directed walk in $D\backslash Z$ with first and last vertex in S that uses a vertex of $D\backslash S$. Now in Section 4 all occurences of Z-normal should be replaced by Z-regular. As a result of this change, "$k+w$" should be replaced by "$k+2w$" in the definition of limited linkage and thereafter, including the running time estimates. In the proof of (4.5) $j-1$ should be changed to $\lfloor j/2 \rfloor$.

The second difficulty concerns the assertion made in the description of (4.4) that "we may assume that the edges $e_1, e_2, \ldots, e_k$ are numbered in such a way that for i, j with $1 \le i < j \le d$ no edge of D has its head in S_i and tail in S_j". It seems that in order to make this assertion we need to convert the arboreal decomposition into a triple (R, X, W) of the same width that satisfies (D1'), (D2") and (D3) below. A proof that it can be done follows. As a result of the bound in (D3), one should be added to the exponent in the estimates on the running times.

An *arboreal predecomposition* of a digraph D is a triple (R, X, W), where R is an arborescence, and $X = (X_e : e \in E(R))$ and $W = (W_r : r \in V(R))$ are sets of vertices of D that satisfy

(D1') $(W_r : r \in V(R))$ is a partition of $V(D)$ into possibly empty sets such that $W_{r_0} \neq \emptyset$, where r_0 is the root of R, and

(D2') if $e \in E(R)$, then $\bigcup\{W_r : r \in V(R), r > e\}$ is X_e-normal or empty.

The *width* of (R, X, W) is the least integer w such that for all $r \in V(R)$, $|W_r \cup \bigcup_{e \sim r} X_e| \le w+1$.

THEOREM. *If D is a digraph of tree-width d, then D has an arboreal predecomposition (R, X, W) of width at most d satisfying*

(D2") if $e \in E(R)$, then $\bigcup\{W_r : r \in V(R), r > e\}$ is the vertex-set of a strongly connected component of $D\backslash X_e$, and

(D3) $|V(R)| \le |V(D)|^2$.

Proof. We say that an arboreal predecomposition (R, X, W) of a digraph D is *tame* if there exists a function $\phi : V(R) \to V(R)$, called a *rank function*, such that for every $r \in V(R)$

(i) $W_{\phi(r)} \neq \emptyset$, and

(ii) r and $\phi(r)$ are at the same distance from the root of R.

Every arboreal decomposition is tame, because the identity is a rank function. Let (R, X, W) be an arboreal predecomposition of D. We say an edge $e \in E(R)$ is *good* if $\bigcup\{W_r : r \in V(R), r > e\}$ is empty or the vertex-set of a strongly connected component of $D\backslash X_e$, and we say that e is *bad* otherwise.

Let (R, X, W) be a tame arboreal predecomposition of D of width at most d with the minimum number of bad edges, and, subject to that, with the minimum number of vertices. Such a choice is possible, because every arboreal decomposition is tame. We claim that (R, X, W) satisfies the conclusion of the theorem.

To prove that (R, X, W) satisfies (D2") we first prove that (R, X, W) has no bad edges. Suppose for a contradiction that (R, X, W) has a bad edge, and choose a bad edge e_1 such that no bad edge has both ends $> e_1$ in R. Let r_0 be the tail and r_1 the head of e_1. Let $S_1, S_2, \ldots, S_k$ be the vertex-sets of the strong components of $D \backslash X_{e_1}$ comprising $\bigcup\{W_r : r \in V(R), r > e_1\}$. Then $k \geq 2$, because e_1 is bad. Let R_1 be the subarborescence of R consisting of r_1 and all vertices $> r_1$ in R, let $R_2, R_3, \ldots, R_k$ be isomorphic copies of R_1 and let their roots be $r_2, r_3, \ldots, r_k$, respectively. Let R' be obtained from $R \cup R_2 \cup R_3 \cup \cdots \cup R_k$ by adding, for each $i = 2, 3, \ldots, k$, an edge e_i with tail r_0 and head r_i. For $f \in E(R')$ we put $X'_f = X_e$, where $e \in E(R)$ is such that $f = e$ or f is a copy of e. For $r \in V(R_i)$ we put $W'_r = W_r \cap S_i$ and for all other $r \in V(R)$ we put $W'_r = W_r$. Let $X' = (X'_f : f \in E(R'))$ and $W' = (W'_r : r \in V(R'))$.

We claim that (R', X', W') is an arboreal predecomposition of D of width at most d. Condition (D1') and the statement about width clearly hold. The edges $e_1, e_2, \ldots, e_k$ are clearly good, and for edges $e \in E(R)$ not in $R_i \cup \{e_i\}$ for any i the set $\bigcup\{W_r : r \in V(R), r > e\}$ has not changed, and hence those edges satisfy (D2') and they are good in (R, X, W) if and only if they are good in (R', X', W'). We now prove that for $i = 1, 2, \ldots, k$ every edge $e \in E(R_i)$ is good. To this end let $C' := \bigcup\{W'_r : r \in V(R'), r > e\}$ be non-empty, and let $f \in E(R_1)$ be such that e is a copy of f. Then $C := \bigcup\{W_r : r \in V(R), r > f\}$ is the vertex-set of a strong component of $D \backslash X_f$, because f if good for (R, X, W) by the choice of e_1. The set C is a subset of $\bigcup\{W_r : r \in V(R), r > e_1\}$, but the latter set is disjoint from X_{e_1} by (D2'). Thus $C \cap X_{e_1} = \emptyset$, and hence C is a subset of the vertex-set of a component of $D \backslash X_{e_1}$. But C intersects S_i (because C' is not empty), and hence $C \subseteq S_i$. Thus $C' = C \cap S_i = C$ is the vertex-set of a strong component of $D \backslash X_f = D \backslash X'_e$, and hence e is good. Thus (R', X', W') is an arboreal predecomposition of D with fewer bad edges than (R, X, W).

Let us now construct a rank function for (R', X', W'). Let $p : V(R') \to V(R)$ be the natural projection. If $W'_{\phi(p(r))} \neq \emptyset$ we define $\phi'(r) = \phi(p(r))$. Otherwise $\phi(p(r)) \in V(R_1)$ and hence $W'_{\phi(p(r))} \cap S_i \neq \emptyset$ for some $i = 1, 2, \ldots, k$. We define $\phi'(r)$ to be the copy of $\phi(p(r))$ in R_i. The function ϕ' thus defined is a rank function for (R', X', W'), proving that (R', X', W') is tame. That is a contradiction, because (R', X', W') has fewer bad edges than (R, X, W). This proves that (R, X, W) has no bad edges. It follows that (R, X, W) satisfies (D2"), for otherwise R has a leaf r with $W_r = \emptyset$, and deleting r violates the minimality of R.

To prove (D3) we choose, for every $r \in V(R)$, a vertex $\alpha(r) \in W_{\phi(r)}$ and a vertex $\beta(r) \in W_{r'}$ for some r' such that $r' = r$ or $r' > r$ in R. We claim that the mapping $r \to (\alpha(r), \beta(r))$ is one-to-one. To prove this suppose that $\alpha(r_1) = \alpha(r_2)$ and $\beta(r_1) = \beta(r_2)$. The former equality implies that $\phi(r_1) = \phi(r_2)$. The latter equality implies that there exists a vertex $r' \in V(R)$ such that for $i = 1, 2$ either $r' = r_i$ or $r' > r_i$ in R. But condition (ii) in the definition of tameness implies $r_1 = r_2$, as desired. This proves (D3) and hence the theorem.