EXCLUDING SUBDIVISIONS OF BOUNDED DEGREE GRAPHS[1]

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Abstract

Let H be a fixed graph. What can be said about graphs G that have no subgraph isomorphic to a subdivision of H? Grohe and Marx proved that such graphs G satisfy a certain structure theorem that is not satisfied by graphs that contain a subdivision of a (larger) graph H_1. Dvořák found a clever strengthening—his structure is not satisfied by graphs that contain a subdivision of a graph H_2, where H_2 has "similar embedding properties" as H. Building upon Dvořák's theorem, we prove that said graphs G satisfy a similar structure theorem. Our structure is not satisfied by graphs that contain a subdivision of a graph H_3 that has similar embedding properties as H and has the same maximum degree as H. This will be important in a forthcoming application to well-quasi-ordering.

1 Introduction

In this paper *graphs* are finite and are permitted to have loops and parallel edges. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. The cornerstone of the Graph Minors project of Robertson and Seymour is the following excluded minor theorem. (The missing definitions are as in [15] and are given at the end of this section.)

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Theorem 1.1 ([15, Theorem (1.3)]). *Let L be a graph. Then there exist integers $\kappa, \rho, \xi > 0$ such that every graph G with no L-minor can be constructed by clique-sums, starting from graphs that are an $\leq \xi$-extension of an outgrowth by $\leq \kappa$ ρ-rings of a graph that can be drawn in a surface in which L cannot be drawn.*

In this paper we are concerned with excluding topological minors. The first such theorem was obtained by Grohe and Marx.

Theorem 1.2 ([3, Corollary 4.4]). *For every graph H there exist integers ξ, κ, ρ, g, D such that every graph G with no H-subdivision can be constructed by clique-sums, starting from graphs that are an $\leq \xi$-extension of either*

(a) a graph of maximum degree D, or

(b) an outgrowth by $\leq \kappa$ ρ-rings of a graph that can be drawn in a surface of genus at most g.

Thus the second outcome includes graphs drawn on surfaces in which H *can* be drawn. Dvořák [1, Theorem 3] strengthened the result by restricting the graphs in (b) to those that can be drawn in a surface Σ in which H can possibly be drawn, but only "in a way in which H cannot be drawn in Σ". We omit the precise statement of Dvořák's theorem, because it requires a large amount of definitions that we otherwise do not need. Instead, let us remark that the meaning of "the way in which H cannot be drawn in Σ" has to do with the function mf, defined as follows.

Let H be a graph and Σ a surface in which H can be embedded. We define $\mathrm{mf}(H, \Sigma)$ as the minimum of $|S|$, over all embeddings of H in Σ and all sets S of regions of the embedded graph such that every vertex of H of degree at least four is incident with a region in S. When H cannot be embedded in Σ, we define $\mathrm{mf}(H, \Sigma)$ to be infinity.

Our objective is to strengthen the theorems of Grohe and Marx, and Dvořák by reducing the value of the constant D to the maximum degree of H, which is clearly best possible. However, we are not able to extend the theorems verbatim; our theorem gives a structure relative to a tangle, as follows. (Tangles, vortices and segregations are defined in Section 2.)

Theorem 1.3. *Let $d \geq 4$ and $h > 0$ be integers. Then there exist integers $\theta, \kappa, \rho, \xi, g \geq 0$ such that the following holds. If H is a graph of maximum degree d on h vertices, and a graph G does not admit an H-subdivision, then* *for every tangle $\mathcal{T}$ in G of order at least θ there exists a set $Z \subseteq V(G)$ with $|Z| \leq \xi$ such that either*

1. *for every vertex $v \in V(G) - Z$ there exists $(A, B) \in \mathcal{T} - Z$ of order at most $d - 1$ such that $v \in V(A) - V(B)$, or*
2. *there exists a $(\mathcal{T} - Z)$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G - Z$ with $|\mathcal{S}_2| \leq \kappa$ such that $\mathcal{S}$ has a proper arrangement in some surface Σ of genus at most g, every society (S_1, Ω_1) in $\mathcal{S}_1$ satisfies $|\overline{\Omega_1}| \leq 3$, every society (S_2, Ω_2) in $\mathcal{S}_2$ is a ρ-vortex, and either*
   - *(a) H cannot be drawn in Σ, or*
   - *(b) H can be drawn in Σ and $\mathrm{mf}(H, \Sigma) \geq 2$, and there exists $\mathcal{S}'_2 \subseteq \mathcal{S}_2$ with $|\mathcal{S}'_2| \leq \mathrm{mf}(H, \Sigma) - 1$ such that for every vertex $v \in V(G) - Z$ either $v \in V(S) - \bar{\Omega}$ for some $(S, \Omega) \in \mathcal{S}'_2$ or there exists $(A, B) \in \mathcal{T} - Z$ of order at most $d - 1$ such that $v \in V(A) - V(B)$.*

Theorem 1.3 has the following immediate corollary.

Corollary 1.4. *Let $d \geq 4$ and $h > 0$ be integers. Then there exist θ and ξ such that for every graph H of order h and of maximum degree d that can be drawn in the plane such that every vertex of degree at least four is incident with the infinite region, and for every graph G, either G admits an H-subdivision, or for every tangle $\mathcal{T}$ of order at least θ in G, there exists $Z \subseteq V(G)$ with $|Z| \leq \xi$ such that for every vertex $v \in V(G) - Z$ there exists $(A, B) \in \mathcal{T} - Z$ of order at most $d - 1$ such that $v \in V(A) - V(B)$.*

Proof. Let $d \geq 4$ and h be given, let θ and ξ be as in Theorem 1.3, and let H be as in the statement of the corollary. Then $\mathrm{mf}(H, \Sigma) = 1$ for every surface Σ, and hence the second outcome of Theorem 1.3 cannot hold. Thus the first outcome holds, as desired. ■

Corollary 1.4 will be used in [6] to prove the following theorem, conjectured by Robertson. In the application it will be important that the order of the separation in Corollary 1.4 is at most $d - 1$.

Theorem 1.5. *Let $k \geq 1$ be an integer, let R denote the graph obtained from a path of length k by replacing each edge by a pair of parallel edges, and let $G_1, G_2, \ldots$ be an infinite sequence of graphs such that none of them has an R-subdivision. Then there exist integers i, j such that $1 \leq i < j$ and G_j has a G_i-subdivision.*

Let us now introduce the missing definitions. Given a subset X of the vertex-set $V(G)$ of a graph G, the subgraph of G induced by X is denoted by $G[X]$. We say that a graph G is the *clique-sum* of graphs G_1, G_2 if there exist $V_1 = \{v_{1,1}, ..., v_{1,|V_1|}\} \subseteq V(G_1), V_2 = \{v_{2,1}, v_{2,2}, ..., v_{2,|V_2|}\} \subseteq V(G_2)$ with $|V_1| = |V_2|$ such that $G_1[V_1]$ and $G_2[V_2]$ are complete graphs, and G can be obtained from $G_1 \cup G_2$ by identifying $v_{1,i}$ and $v_{2,i}$ for each i and deleting a subset of edges with both ends in $V_1 \cup V_2$. A graph G' is a *$\leq r$-extension* of a graph G if G can be obtained from G' by deleting at most r vertices of G. A graph G is an *r-ring with perimeter* $t_1, ..., t_n$ if $t_1, ..., t_n \in V(G)$ are distinct and there is a sequence $X_1, ..., X_n$ of subsets of $V(G)$ such that

- $X_1 \cup ... \cup X_n = V(G)$, and every edge of G has both ends in some X_i,
- $t_i \in X_i$ for $1 \leq i \leq n$,
- $X_i \cap X_k \subseteq X_j$ for $1 \leq i \leq j \leq k \leq n$,
- $|X_i| \leq r$ for $1 \leq i \leq n$.

Let G_0 be a graph drawn in a surface Σ, and let $\Delta_1, ..., \Delta_d \subseteq \Sigma$ be pairwise disjoint closed disks, each meeting the drawing only in vertices of G_0, and each containing no vertices of G_0 in its interior. For $1 \leq i \leq d$, let the vertices of G_0 in the boundary of Δ_i be $t_1, ..., t_n$ say, in order, and choose an r-ring G_i with perimeter $t_1, ..., t_n$ meeting G_0 just in $t_1, ..., t_n$ and disjoint from every other G_j; and let G be the union of $G_0, G_1, ..., G_d$. We call such a graph G an *outgrowth by d r-rings of* G_0.

The paper is organized as follows. In Section 2 we review the notions of tangles and graph minors. In Section 3 we prove an Erdős-Pósa-type result for "spiders", trees with one vertex of degree d and all other vertices of degree one or two. In Section 4 we prove a lemma that will allow us to find a large well-behaved family of spiders, given a huge number of spiders. In Section 5 we review some theorems related to graphs embedded on a surface, and prove some other lemmas. In Section 6 we prove Theorem 1.3.

We remark that our proof of Theorem 1.3 uses the Graph Minors theory developed by Robertson and Seymour and is inspired by ideas in the proof of Dvořák's theorem [1]. We would like to acknowledge that we have benefited from conversations with Paul Wollan, and that the paper is based on part of the PhD dissertation [5] of the first author.

2 Tangles and minors

In this section, we review some theorems about tangles and graph minors.

A *separation* of a graph G is a pair (A, B) of subgraphs with $A \cup B = G$ and $E(A \cap B) = \emptyset$, and the *order* of (A, B) is $|V(A) \cap V(B)|$. A *tangle* $\mathcal{T}$ in G of *order* θ is a set of separations of G, each of order less than θ such that

(T1) for every separation (A, B) of G of order less than θ, either $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$;

(T2) if $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$, then $A_1 \cup A_2 \cup A_3 \neq G$;

(T3) if $(A, B) \in \mathcal{T}$, then $V(A) \neq V(G)$.

The notion of tangle was first defined by Roberson and Seymour in [10]. (T1), (T2) and (T3) are called the first, second and third tangle axiom, respectively.

Given a graph H, an *H-minor* of a graph G is a map α with domain $V(H) \cup E(H)$ such that the following hold.

- $\alpha(h)$ is a nonempty connected subgraph of G, for every $h \in V(H)$.
- If h_1 and h_2 are different vertices of H, then $\alpha(h_1)$ and $\alpha(h_2)$ are disjoint.
- For each edge e of H with ends h_1, h_2, $\alpha(e)$ is an edge of G with one end in $\alpha(h_1)$ and one end in $\alpha(h_2)$; furthermore, if $h_1 = h_2$, then $\alpha(e) \in E(G) - E(\alpha(h_1))$.
- If e_1, e_2 are two different edges of H, then $\alpha(e_1) \neq \alpha(e_2)$.

We say that G *contains an H-minor* if such a function α exists. For every $h \in V(H)$, $\alpha(h)$ is called a *branch set* of α. A tangle $\mathcal{T}$ in G *controls* an H-minor α if α is an H-minor such that there does not exist $(A, B) \in \mathcal{T}$ of order less than $|V(H)|$ and $h \in V(H)$ such that $V(\alpha(h)) \subseteq V(A)$.

The following theorem offers a way to obtain a tangle in a graph from a minor.

Theorem 2.1 ([10, Theorem (6.1)])**.** *Let G and H be graphs. Let $\mathcal{T}'$ be a tangle in H of order $\theta \geq 2$. If G admits an H-minor, and $\mathcal{T}$ is the set of separations (A, B) of G of order less than θ such that there exists $(A', B') \in \mathcal{T}'$ with $E(A') = E(A) \cap \alpha(E(H))$, then $\mathcal{T}$ is a tangle in G of order θ.*

The tangle $\mathcal{T}$ in Theorem 2.1 is called the *tangle induced by* $\mathcal{T}'$. We say that $\mathcal{T}'$ is *conformal* with a tangle $\mathcal{T}''$ in G if $\mathcal{T} \subseteq \mathcal{T}''$.

A *society* is a pair (S, Ω), where S is a graph and Ω is a cyclic permutation of a subset $\bar{\Omega}$ of $V(S)$. Let ρ be a nonnegative integer. A society (S, Ω) is a ρ*-vortex* if for all distinct $u, v \in \bar{\Omega}$, there do not exist $\rho + 1$ mutually disjoint paths of S between $I \cup \{u\}$ and $J \cup \{v\}$, where I is the set of vertices in $\bar{\Omega}$ after u and before v in the natural order, and J is the set of vertices in $\bar{\Omega}$ after v and before u.

A *segregation* of a graph G is a set $\mathcal{S}$ of societies such that the following hold.

- S is a subgraph of G for every $(S, \Omega) \in \mathcal{S}$, and $\bigcup\{S : (S, \Omega) \in \mathcal{S}\} = G$.
- For every distinct (S, Ω) and $(S', \Omega') \in \mathcal{S}$, $V(S \cap S') \subseteq \bar{\Omega} \cap \overline{\Omega}'$ and $E(S \cap S') = \emptyset$.

We write $V(\mathcal{S}) = \bigcup\{\bar{\Omega} : (S, \Omega) \in \mathcal{S}\}$. If $\mathcal{T}$ is a tangle in G, a segregation $\mathcal{S}$ of G is $\mathcal{T}$*-central* if for every $(S, \Omega) \in \mathcal{S}$, there is no $(A, B) \in \mathcal{T}$ of order at most half of the order of $\mathcal{T}$ with $B \subseteq S$.

A *surface* is a nonnull compact connected 2-manifold without boundary. Let Σ be a surface and $\mathcal{S} = \{(S_1, \Omega_1), ..., (S_k, \Omega_k)\}$ a segregation of G. An *arrangement* of $\mathcal{S}$ in Σ is a function α with domain $\mathcal{S} \cup V(\mathcal{S})$, such that the following hold.

- For $1 \leq i \leq k$, $\alpha(S_i, \Omega_i)$ is a closed disk $\Delta_i \subseteq \Sigma$, and $\alpha(x) \in \partial\Delta_i$ for each $x \in \overline{\Omega_i}$.
- For $1 \leq i \leq k$, if $x \in \Delta_i \cap \Delta_j$, then $x = \alpha(v)$ for some $v \in \overline{\Omega_i} \cap \overline{\Omega_j}$.
- For all distinct $x, y \in V(\mathcal{S})$, $\alpha(x) \neq \alpha(y)$.
- For $1 \leq i \leq k$, Ω_i is mapped by α to the natural order of $\alpha(\overline{\Omega_i})$ determined by $\partial\Delta_i$.

An arrangement is *proper* if $\Delta_i \cap \Delta_j = \emptyset$ for all $1 \leq i < j \leq k$ such that $|\overline{\Omega_i}|, |\overline{\Omega_j}| > 3$.

Given a graph H, an H*-subdivision* is a pair of functions (π_V, π_E) such that the following hold.

- $\pi_V : V(H) \to V(G)$ is an injective function.

- π_E maps loops of H to cycles in G and maps other edges of H to paths in G such that $\pi_E(e)$ contains $\pi_V(v)$, and $\pi_E(e')$ has the ends $\pi_V(x)$ and $\pi_V(y)$ for every loop e with end v and every edge $e = xy \in E(H)$.

- If f_1, f_2 are two different edges in H, then $\pi_E(f_1)$ and $\pi_E(f_2)$ are internally vertex-disjoint.

We say that G *admits an H-subdivision* if such a pair of functions (π_V, π_E) exists.

3 Finding disjoint spiders

First, we introduce a lemma proved by Robertson and Seymour [13].

Lemma 3.1 ([13, Theorem (5.4)])**.** *Let G be a graph, and let Z be a subset of $V(G)$ with $|Z| = \xi$. Let $k \geq \lceil \frac{3}{2}\xi \rceil$, and let α be a K_k-minor in G. If there is no separation (A, B) of G of order less than $|Z|$ such that $Z \subseteq V(A)$ and $A \cap \alpha(h) = \emptyset$ for some $h \in V(K_k)$, then for every partition $(Z_1, ..., Z_n)$ of Z into non-empty subsets, there are n connected graphs $T_1, ..., T_n$ of G, mutually disjoint and $V(T_i) \cap Z = Z_i$ for $1 \leq i \leq n$.*

A *d-spider with head v* is a tree such that every vertex other than v in the tree has degree at most 2, and the degree of v is d. A *leaf* is a vertex of degree one. Let G be a graph, and let S, Y be subsets of $V(G)$. A *d-spider from S to Y* is a d-spider with head $v \in S$ whose leaves are in Y.

Let G be a graph and $\mathcal{T}$ a tangle in G. We say that a subset X of $V(G)$ is *free* if there exists no $(A, B) \in \mathcal{T}$ of order less than $|X|$ such that $X \subseteq V(A)$.

Lemma 3.2. *Let G be a graph and H be a graph on h vertices of maximum degree d. Let $t \geq \lceil \frac{3hd}{2} \rceil$. Let $\mathcal{T}$ be a tangle of order at least hd in G that controls a K_t-minor. If there exist pairwise disjoint sets $X_1, X_2, ..., X_h$ such that for $1 \leq i \leq h$ the set X_i consists of a vertex of G and $d-1$ of its neighbors and $\bigcup_{i=1}^{h} X_i$ is free with respect to $\mathcal{T}$, then G has an H-subdivision.*

Proof. Let $Z = \bigcup_{i=1}^{h} X_i$, and let α be a K_t-minor controlled by $\mathcal{T}$. Suppose that there exists a separation (A, B) of G of order less than $|Z|$ such that $Z \subseteq V(A)$ and $A \cap \alpha(v) = \emptyset$ for some $v \in V(K_t)$. By the first tangle axiom, either $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$. Since Z is free, $(B, A) \in \mathcal{T}$. But it is a contradiction since $t \geq hd$ and $\mathcal{T}$ controls α. Therefore, there does not exist a separation (A, B) of G of order less than $|Z|$ such that $Z \subseteq V(A)$ and $A \cap \alpha(v) = \emptyset$ for some $v \in V(K_t)$.

Denote $V(H)$ by $\{u_1, u_2, ..., u_h\}$ and $E(H)$ by $\{e_1, e_2, ..., e_{|E(H)|}\}$. Since the maximum degree of H is at most d, there exist $Z_0 \subseteq Z$ and a partition $(Z_1, Z_2, ..., Z_{|E(H)|})$ of $Z - Z_0$ such that for every $1 \leq \ell \leq |E(H)|$, Z_ℓ consists of two distinct vertices where one is in X_i and one is in X_j, where the ends of e_ℓ are u_i and u_j. By Lemma 3.1, there exist $|E(H)|$ pairwise disjoint paths in $G' - Z_0$ connecting the two vertices of each part of $(Z_1, Z_2, ..., Z_{|E(H)|})$. This creates a subdivision of H. ■

Theorem 3.3 ([8, Theorem 6]). *Let G be a graph and $\mathcal{T}$ a tangle in G of order θ. Let $\{X_j \subseteq V(G) : j \in J\}$ be a family of subsets of $V(G)$ indexed by J. Let d, k be an integer with $\theta \geq (k+d)^{d+1} + d$. If $|X_j| = d$ for every $j \in J$, then there exists a set $J' \subseteq J$ satisfying the following.*

1. *For all $j \neq j' \in J'$, X_j and $X_{j'}$ are disjoint.*
2. *$\bigcup_{j \in J'} X_j$ is free.*
3. *If $|\bigcup_{j \in J'} X_j| \leq k$, then there exists Z with $|Z| \leq (k+d)^{d+1}$ satisfying that for all $j \in J'$, either $X_j \cap Z \neq \emptyset$, or X_j is not free in $\mathcal{T} - Z$.*

Theorem 3.4. *Let h and d be positive integers. Let G be a graph, and let S be a subset of vertices of degree at least d in G. Let $\mathcal{T}$ be a tangle in G of order θ. If $\theta \geq (hd)^{d+1} + d$, then either*

1. *there exist h vertices $v_1, v_2, ..., v_h \in S$ and h pairwise disjoint subsets $X_1, X_2, ..., X_h$ of $V(G)$, where X_i consists of v_i and $d-1$ neighbors of v_i for each $1 \leq i \leq h$, such that $\bigcup_{i=1}^{h} X_i$ is free in $\mathcal{T}$, or*
2. *there exists a set $C \subseteq V(G)$ with $|C| \leq (hd)^{d+1}$ such that for every $v \in S - C$, there exists $(A, B) \in \mathcal{T} - C$ of order less than d such that $v \in V(A) - V(B)$.*

Proof. Let $\{X_j : j \in J\}$ be the collection of the d-element subsets consisting of one vertex v_j in S and $d-1$ of its neighbors. Applying Theorem 3.3 by further taking $k = (h-1)d$, then there exists $J' \subseteq J$ such that $X_j \cap X_{j'} = \emptyset$ for every distinct j, j' in J', and $\bigcup_{j \in J'} X_j$ is free. Furthermore, if $|\bigcup_{j \in J'} X_j| \leq (h-1)d$, there exists $C \subseteq V(G)$ with $|C| \leq (hd)^{d+1}$ satisfying that for all $j \in J'$, either $X_j \cap C \neq \emptyset$, or X_j is not free in $\mathcal{T} - C$.

Observe that if $|\bigcup_{j\in J'} X_j| > (h-1)d$, then $|J'| \geq h$ and the first statement holds. So we assume that $|\bigcup_{j\in J'} X_j| \leq (h-1)d$, and we shall prove that the second statement of this theorem holds. Let $v \in S - C$. Suppose that there does not exist $(A, B) \in \mathcal{T} - C$ of order less than d such that $v \in V(A) - V(B)$. Let U be the collection of those X_j that is disjoint from C and consists of v and $d-1$ neighbors of v. For every member X_j of U, we define the *rank* of X_j to be the minimum order of a separation $(A, B) \in \mathcal{T} - C$ such that $X_j \subseteq V(A)$. As none of member of U is free, the rank of each member of U is at most $d-1$. Let r be the maximum rank of a member of X_j, and let X be a member of U of rank r. Let $(A, B) \in \mathcal{T} - C$ of order r such that $X \subseteq V(A)$, and subject to that, $|V(B) - V(A)|$ is as small as possible. By the assumption, $v \in V(A) \cap V(B)$ and $r \leq d-1$. On the other hand, there exist r disjoint paths from $X - \{v\}$ to $V(B)$, as v is adjacent to all vertices in $X - \{v\}$. We denote these r disjoint paths by $P_1, P_2, ..., P_r$, and denote the end of P_i in $X - \{v\}$ by u_i for $1 \leq i \leq r$. As $v \in V(A) \cap V(B)$ and $|V(A) \cap V(B)| = r$, $v \in V(P_i)$ for some $1 \leq i \leq r$. Without loss of generality, we may assume that $v \in V(P_r)$. In addition, v is adjacent to a vertex u in $V(B) - V(A)$, otherwise, the rank of X is smaller than r. As $(X - \{u_r\}) \cup \{u\}$ is a member of U, its rank is at most r. Let $(A', B') \in \mathcal{T} - C$ be a separation of order at most r such that $(X - \{u_r\}) \cup \{u\} \subseteq V(A')$. $X \subseteq V(A \cup A')$ and $u \in (V(B) - V(A)) - (V(B \cap B') - V(A \cup A'))$, so the order of $(A \cup A', B \cap B')$ is at least $r+1$ by the choice of (A, B). It implies that the order of $(A \cap A', B \cup B')$ is at most $r-1$. Notice that $v \in V(A') \cap V(B')$ by the assumption, so $((A \cap A') - \{v\}, (B \cup B') - \{v\})$ is a separation of $G - \{v\}$ of order less than $r-1$. But $P_1, P_2, ..., P_{r-1}$ are $r-1$ disjoint paths from $V(A \cap A') - \{v\}$ to $V(B \cup B') - \{v\}$ in $G - \{v\}$, a contradiction. This proves the second statement. ■

We need the following variation of Theorem 3.4. A version for edge-disjoint spiders was proved in [7] and [8, Theorem 6].

Theorem 3.5. *Let G be a graph, and let X, Y be disjoint subsets of $V(G)$. Let h, d be nonnegative integers. Then either there exist h disjoint d-spiders from X to Y, or there exists $C \subseteq V(G)$ with $|C| \leq \frac{3}{2}(hd)^{d+1} + \frac{d}{2} + 1$ such that every d-spider from X to Y intersects C.*

Proof. Note that for every subset C of Y such that $|Y - C| \leq d-1$, every d-spider from X to C intersects C. So we may assume that $|Y| \geq \frac{3}{2}((hd)^{d+1} + d)$, otherwise we are done. Let G' be the graph obtained from G by adding edges such that Y induced a clique in G'. As every clique of size k contains a tangle of order $\lfloor 2k/3 \rfloor$, $G'[Y]$ contains a tangle of order $(hd)^{d+1}+d$. And Y is a minor of G', so G' contains a tangle $\mathcal{T}$ of order $(hd)^{d+1}+d$ induced by $G'[Y]$ by Theorem 2.1 such that $Y \subseteq V(B)$ for every $(A,B) \in \mathcal{T}$. Let $\{X_j : j \in J\}$ be the collection of d-element subsets of $V(G)$ such that every X_j consisting of one vertex x in X and $d-1$ neighbors of x. By Theorem 3.3, there exists $J' \subseteq J$ such that $X_j \cap X_{j'} = \emptyset$ for every distinct j, j' in J', and $\bigcup_{j \in J'} X_j$ is free. Furthermore, if $|\bigcup_{j \in J'} X_j| \leq (h-1)d$, there exists $C \subseteq V(G)$ with $|C| \leq (hd)^{d+1}$ satisfying that for all $j \in J'$, either $X_j \cap C \neq \emptyset$, or X_j is not free in $\mathcal{T} - Z$.

First, assume that $|\bigcup_{j \in J'} X_j| > (h-1)d$, so $|J'| \geq h$. Let $\{1, 2, ..., h\} \subseteq J'$, and let x_j be a vertex in $X_j \cap X$ for $1 \leq j \leq h$. Suppose that there do not exist dh disjoint paths from $\bigcup_{j=1}^h X_j$ to Y in G'. Then there exists a separation (A,B) of G' of order less than dh such that $\bigcup_{j=1}^h X_j \subseteq V(A)$ and $Y \subseteq V(B)$. Since $Y \subseteq V(B)$, we know that $(A,B) \in \mathcal{T}$. But it implies that $\bigcup_{j=1}^h X_j$ is not free, a contradiction. Hence, there exist dh disjoint paths from $\bigcup_{j=1}^h X_j$ in G'. That is, there exist h disjoint d-spiders from x_j to Y in G'. We are done in this case since every d-spiders from X to Y in G' contains a d-spider from X to Y in G as a subgraph.

So we may assume that $|\bigcup_{j \in J'} X_j| \leq (h-1)d$, there exists $C \subseteq V(G)$ with $|C| \leq (hd)^{d+1}$ satisfying that for all $j \in J'$, either $X_j \cap C \neq \emptyset$, or X_j is not free in $\mathcal{T} - C$. Let $v \in V(G) - C$, and let D be a d-spider from v to Y in G. Note that D is also a d-spider from v to Y in G'. Suppose that D is disjoint from C. So D contains some X_j such that $v \in X_j$ and $X_j \cap C = \emptyset$. Since X_j is not free in $\mathcal{T} - C$, there exists $(A,B) \in \mathcal{T} - C$ of order less than d such that $X_j \subseteq V(A)$ and $Y - C \subseteq V(B)$. It is a contradiction since there exist d disjoint paths in D from $V(A)$ to $V(B)$. This proves that D intersects C. ■

4 Taming spiders

We say that (S, Ω, Ω_0) is a *neighborhood* if S is a graph and Ω, Ω_0 are cyclic permutations with $\overline{\Omega}, \overline{\Omega_0} \subseteq V(S)$. A neighborhood (S, Ω, Ω_0) is *rural* if S has a drawing Γ in the plane without crossings and there are disks $\Delta_0 \subseteq \Delta$ such that

- Γ uses no point outside Δ and none in the interior of Δ_0, and

- $\bar{\Omega}$ are the vertices in $\Gamma \cap \partial\Delta$, and $\overline{\Omega_0}$ are the vertices in $\Gamma \cap \Delta_0$, and
- the cyclic permutations of $\bar{\Omega}$ and $\overline{\Omega_0}$ coincide with the natural cyclic order on Δ and Δ_0.

In this case, we say that $(\Gamma, \Delta, \Delta_0)$ is a *presentation* of (S, Ω, Ω_0). For a fixed presentation $(\Gamma, \Delta, \Delta_0)$ of a neighborhood (S, Ω, Ω_0) and an integer $s \geq 0$, an *s-nest* for $(\Gamma, \Delta, \Delta_0)$ is a sequence $(C_1, C_2, ..., C_s)$ of pairwise disjoint cycles of S such that $\Delta_0 \subseteq \Delta_1 \subseteq ... \subseteq \Delta_s \subseteq \Delta$, where Δ_i is the closed disk in the plane bounded by C_i.

If (S, Ω, Ω_0) is a neighborhood and (S_0, Ω_0) is a society, then $(S \cup S_0, \Omega)$ is a society and we call this society the *composition* of the society (S_0, Ω_0) with the neighborhood (S, Ω, Ω_0). A society (S, Ω) is *s-nested* if it is the composition of a society with a rural neighborhood that has an s-nest for some presentation of it.

A subgraph F of a rural neighborhood (S, Ω, Ω_0) is *perpendicular* to an s-nest $(C_1, C_2, ..., C_s)$ if for every component P of F

- P is a path with one end in $\bar{\Omega}$ and the other in $\overline{\Omega_0}$, and
- $P \cap C_i$ is a path for all $i = 1, 2, ..., s$.

We shall use the following theorem, which was proved in [4], to prove the main theorem of this section. We present a simplified restatement of it.

Theorem 4.1 ([4, Theorem 10.3])**.** *For every three positive integers s, k, c, there exists an integer $s'(s, k, c)$ such that for every s'-nested society (S, Ω) that is a composition of a society (S_0, Ω_0) with a rural neighborhood with a s'-nest, and for every union of c pairwise disjoint k-spiders F_0 from $V(S_0) - \overline{\Omega_0}$ to $\bar{\Omega}$, there exists a union of c pairwise disjoint k-spiders F in (S, Ω) from the set of the heads of F_0 to the set of leaves of F_0 such that (S, Ω) can be expressed as a composition of some society with a rural neighborhood (S', Ω, Ω') that has a presentation with an s-nest $(C_1, C_2, ..., C_s)$ such that $S' \cap F$ is perpendicular to $(C_1, C_2, ..., C_s)$.*

Now, we are ready to prove the main theorem of this section.

Theorem 4.2. *For every positive integers $d \geq 3, \rho, k$ and s, there exist integers $s' = s'(k, d, s, \rho)$ and $k' = k'(k, d, \rho)$ such that for every s'-nested society (S, Ω) that is a composition of a ρ-vortex (S_0, Ω_0) with a rural neighborhood* *that has an s'-nest, and for every k' pairwise disjoint d-spiders $D_1, D_2, ..., D_{k'}$ from $V(S_0) - \overline{\Omega_0}$ to $\bar{\Omega}$, there exist k pairwise disjoint d-spiders $D'_1, D'_2, ..., D'_k$ from $V(S_0)$ to $\bar{\Omega}$ such that the following hold.*

1. *(S, Ω) can be expressed as a composition of a society (S'_0, Ω') with a rural neighborhood (S', Ω, Ω') that has a presentation with an s-nest $(C_1, C_2, ..., C_s)$ such that $D'_i \cap S'$ is perpendicular to $(C_1, C_2, ..., C_s)$ for every $1 \leq i \leq k$.*

2. *For every $1 \leq i \leq k$, the head of D'_i is the head of $D_{i'}$ for some $1 \leq i' \leq k'$.*

3. *For every $1 \leq i \leq k$, every leaf of D'_i is a leaf of $D_1 \cup D_2 \cup ... \cup D_{k'}$.*

4. *For every $1 \leq i \leq k$, there exists an interval I_i of $\bar{\Omega}$ containing all leaves of D'_i such that I_i is disjoint from I_j for every $j \neq i$.*

Proof. Let $s'(k, d, s, \rho) = s'_{4.1}(s, d, 3k(\rho + 1))$ and $k'(k, d, \rho) = 3k(\rho + 1)$, where $s'_{4.1}$ is the function s' mentioned in Theorem 4.1. By Theorem 4.1, there exist $3k(\rho+1)$ pairwise disjoint d-spiders $D'_1, D'_2, ..., D'_{k'}$ from the set of the heads of $D_1, D_2, ..., D_{k'}$ to the union of the set of leaves of $D_1, D_2, ..., D_{k'}$ such that (S, Ω) can be expressed as a composition of some society with a rural neighborhood (S', Ω, Ω') that has a presentation with an s'-nest $(C_1, C_2, ..., C_s)$ such that $D'_i \cap S'$ is perpendicular to $(C_1, C_2, ..., C_s)$ for every $1 \leq i \leq k$. For every $1 \leq i \leq k'$, let I_i be a minimum interval of $\bar{\Omega}$ containing all leaves of D_i. Then it is sufficient to prove that there exist $1 \leq i_1 < i_2 < ... < i_k \leq k'$ such that are $I_{i_1}, I_{i_2}, ..., I_{i_k}$ are pairwise disjoint. Suppose that there do not exist such k pairwise disjoint intervals. Then the intersection graph of $I_1, I_2, ..., I_{k'}$ does not contain an independent set of size k, so it contains a clique of size at least $k'/(k-1) > 3(\rho+1)$, as every interval graph is perfect.

Let $\overline{\Omega_0} = \{v_1, v_2, ..., v_{|\overline{\Omega_0}|}\}$ in order. Since (S_0, Ω_0) is a ρ-vortex, by Theorem 8.1 in [9], there exists a path-decomposition $(t_1 t_2 ... t_{|\overline{\Omega_0}|}, \mathcal{X})$ of S_0 such that $|X_{t_i} \cap X_{t_j}| \leq \rho$ for every $1 \leq i < j \leq |\overline{\Omega_0}|$ and $v_i \in X_{t_i}$ for every $1 \leq i \leq |\overline{\Omega_0}|$. Since $S - S_0$ is a plane graph, for every $i \neq j$, if I_i intersects I_j, then there exists an integer a such that $D'_i \cap D'_j \cap X_a \cap X_{a+1} \neq \emptyset$. Let G be the graph obtained from S by adding edges such that $G[X_i \cap X_{i+1}]$ is a clique, for every $1 \leq i \leq |\overline{\Omega_0}| - 1$. Recall that the intersection graph of $I_1, I_2, ..., I_{k'}$ has a clique of size at least $3(\rho + 1)$. Therefore, G contains a $K_{3(\rho+1)}$-minor, where each branch set is D'_i for some i. Without loss of generality, we may assume that the branch set of the $K_{3(\rho+1)}$-minor is $D'_1, D'_2, ..., D'_{3(\rho+1)}$.

Observe that $D'_i \cap X_{t_j}$ is connected in G for every $1 \leq i \leq 3(\rho+1)$ and $1 \leq j \leq |\overline{\Omega_0}|$. Let G' be the graph obtained from G by deleting vertices not in $D'_1 \cup D'_2 \cup ... \cup D'_{3(\rho+1)}$ and then contracting each component of $D'_i \cap (S - S_0)$ into a vertex and contracting $D'_i \cap X_{t_j}$ into a vertex, for every $1 \leq i \leq 3(\rho+1)$ and $1 \leq j \leq |\overline{\Omega_0}|$. Note that G' contains a $K_{3(\rho+1)}$-minor, so the tree-width of G' is at least 3ρ. On the other hand, G' can be written as $G_1 \cup G_2$ such that $V(G_1 \cap G_2) \subseteq \overline{\Omega_0}$, and G_1 is an outerplanar graph that can be drawn in the plane such that the vertices of $V(G_1 \cap G_2)$ are in the boundary of a region in order, and G_2 has a path decomposition of width less ρ such that each bag contains a vertex in $V(G_1 \cap G_2)$ in order. By Lemma 8.1 in [2], G' has tree-width less than 3ρ, a contradiction. This proves the theorem. ∎

5 Theorems on surfaces

In this section, we recall some results about graphs embedded in surfaces.

A *surface* is a compact 2-manifold. An *O-arc* is a subset homeomorphic to a circle, and a *line* is a subset homeomorphic to $[0, 1]$. Let Σ be a surface. For every subset Δ of Σ, we denote the closure of Δ by $\bar{\Delta}$, and the boundary of Δ by $\partial\Delta$. A *drawing* Γ in Σ is a pair (U, V), where $V \subseteq U \subseteq \Sigma$, U is closed, V is finite, $U - V$ has only finitely many arc-wise connected components, called *edges*, and for every edge e, either $\bar{e}$ is a line whose set of ends in $\bar{e} \cap V$, or $\bar{e}$ is an O-arc and $|\bar{e} \cap V| = 1$. The components of $\Sigma - U$ are called *regions*. The members of V are called *vertices*. For a drawing $\Gamma = (U, V)$, we write $U = U(\Gamma), V = V(\Gamma)$, and $E(\Gamma), R(\Gamma)$ are defined to be the set of edges and the set of regions, respectively. The sets $\{v\}$, for $v \in V(\Gamma)$, the sets of edges and regions of Γ are called the *atoms* of Γ. If v is a vertex of a drawing Γ and e is an edge or a region of Γ, we say that e is *incident with* v if v is contained in the closure of e. Note that the incidence relation between $V(\Gamma)$ and $E(\Gamma)$ defines a graph, and we say that Γ is a *drawing of* G in Σ if G is defined by this incident relation. In this case, we say that G is *embeddable* in Σ, or G can be *drawn* in Σ. A drawing is 2-*cell* if Σ is connected and every region is an open disk.

Let Γ be a 2-cell drawing in a surface Σ. We say that a drawing K in Σ is a *radial drawing* of Γ if it satisfies the following conditions.

- $U(\Gamma) \cap U(K) = V(\Gamma) \subseteq V(K)$.

- Each region r of Γ contains a unique vertex of K.

- K is a drawing of a bipartite graph, and $(V(\Gamma), V(K) - V(\Gamma))$ is a bipartition of it.

- For every $v \in V(\Gamma)$, the edges of $K \cup \Gamma$ incident with v belong alternately to Γ and to K (in their cyclic order around v).

Let Σ be a surface, and let Γ be a drawing in Σ. A subset Z of Σ is Γ-*normal* if $Z \cap U(\Gamma) \subseteq V(\Gamma)$. If Σ is connected and not a sphere, we say that Γ is θ-*representative* if $|F \cap V(\Gamma)| \geq \theta$ for every non-null-homotopic Γ-normal O-arc F in Σ.

Let Σ be a surface, and let Γ be a drawing of a graph G in Σ. A *tangle* in Γ is a tangle in G. A tangle $\mathcal{T}$ in Γ of order θ is said to be *respectful (towards* Σ if Σ is connected and for every Γ-normal O-arc F in Σ with $|F \cap V(\Gamma)| < \theta$, there is a closed disk $\Delta \subseteq \Sigma$ with $\partial\Delta = F$ such that $(\Gamma \cap \Delta, \Gamma \cap \overline{\Sigma - \Delta}) \in \mathcal{T}$. It is clear that Δ has to be unique, and we write $\Delta = \mathrm{ins}(F)$; the function *ins* is called the *inside function* of $\mathcal{T}$. Assume that Γ is 2-cell, and let K be the radial drawing of Γ. If W is a closed walk of K, we define $K|W$ to be the subdrawing of K formed by the vertices and the edges in W. If the length of W is less than 2θ, then we define $\mathrm{ins}(W)$ to be the union of $U(K|W)$ and $\mathrm{ins}(C)$, taken over all cycles C of $K|W$. For every two atoms a, b of K, define a function $m_{\mathcal{T}}(a, b)$ as follows:

- if $a = b$, then $m_{\mathcal{T}}(a, b) = 0$;

- if $a \neq b$ and $a, b \subseteq \mathrm{ins}(W)$ for some closed walk W of K of length less than 2θ, then $m_{\mathcal{T}}(a, b) = \min \frac{1}{2}|E(W)|$, taking over all such closed walks W;

- otherwise, $m_{\mathcal{T}}(a, b) = \theta$.

Note that K is bipartite, so $m_{\mathcal{T}}$ is integral. In addition, for every atom c of Γ, we define $a(c)$ to be an atom of K such that

- $a(c) = c$ if $c \subseteq V(\Gamma)$;

- $a(c)$ is the region of K including c if c is an edge of Γ;

- $a(c) = \{v\}$, where v is the vertex of K in c, if c is a region of Γ.

For every atoms b, c of Γ, we define $m_{\mathcal{T}}(b,c) = m_{\mathcal{T}}(a(b), a(c))$. The following is a consequence of Theorem 9.1 of [11].

Theorem 5.1. *Let Σ be a surface, and let Γ be a 2-cell drawing of a graph in Σ. If $\mathcal{T}$ is a respectful tangle in Γ, then $m_{\mathcal{T}}$ is a metric on the atoms of G.*

The following theorem is useful.

Theorem 5.2 ([12, Theorem (1.1)])**.** *Let Σ be a surface, and let Γ be a 2-cell drawing of a graph in Σ with $E(\Gamma) \neq \emptyset$. Let $\mathcal{T}$ be a respectful tangle of order θ in Γ, and let K be a radial drawing of Γ. Then $(A, B) \in \mathcal{T}$ if and only if for every edge e of A, there exists a cycle C of K with $V(C) \cap V(\Gamma) \subseteq V(A) \cap V(B)$ and with $e \subseteq \mathrm{ins}(C)$.*

Theorem 5.3. *Let Σ be a surface, and let Γ be a 2-cell drawing of a graph in Σ with $E(\Gamma) \neq \emptyset$. Let $\mathcal{T}$ be a respectful tangle of order θ in Γ. Let $x \in V(\Gamma)$. If $(A, B) \in \mathcal{T}$ is a separation of Γ such that $x \in V(A) - V(B)$ and subject to that, A is minimal, then $m_{\mathcal{T}}(x, y) \leq |V(A) \cap V(B)|$ for every $y \in V(A)$.*

Proof. Let $y \in V(A)$ be a vertex different from x. Since $(A, B) \in \mathcal{T}$ is a separation with the minimal A such that $x \in V(A) - V(B)$, there exists a path P in A from x to y internally disjoint from $V(B)$. Let e be the edge in P incident with x. By Theorem 5.2, there exists a cycle C of the radial drawing K of Γ with $V(C) \cap V(\Gamma) \subseteq V(A) \cap V(B)$ and with $e \subseteq \mathrm{ins}(C)$. So $x \in \mathrm{ins}(C)$. If $y \notin \mathrm{ins}(C)$, then C intersects P at an internal vertex of P. However, $V(C) \cap V(\Gamma) \subseteq V(A) \cap V(B)$. This implies that some internal vertex of P is in $V(A) \cap V(B)$, a contradiction. Hence, $y \in \mathrm{ins}(C)$. Therefore, $m_{\mathcal{T}}(x, y) \leq |V(A) \cap V(B)|$. ■

Theorem 5.4 ([11, Theorem (8.12)], [12, Theorem (1.2)])**.** *Let $\mathcal{T}$ be a respectful tangle of order θ, where $\theta \geq 2$, in a 2-cell drawing Γ in a connected surface Σ. If c is an atom in Γ, then there exists an edge e of Γ such that $m_{\mathcal{T}}(c, e) = \theta$.*

Let Γ be a 2-cell drawing in a surface Σ, and let $\mathcal{T}$ be a respectful tangle of order θ in Γ. Let x be an atom of Γ. A *λ-zone around x* is an open disk Δ in Σ with $x \subseteq \Delta$, such that $\partial\Delta$ is an O-arc, $\partial\Delta \subseteq \Gamma$, $m_{\mathcal{T}}(x, y) \leq \lambda$ for every atom y of G with $y \subseteq \bar{\Delta}$, and if $x \in E(G)$, then $\lambda \geq 2$. A *λ-zone* is a λ-zone around some atom.

Let Δ be a λ-zone. Note that $U(\Gamma) \cap \partial\Delta$ is a cycle, and the drawing $\Gamma' = \Gamma \cap (\Sigma - \Delta)$ is 2-cell in Σ. We say that Γ' is the *drawing obtained from* Γ *by clearing* Δ. We say that $\mathcal{T}'$ is a *tangle of order* $\theta - 4\lambda - 2$ *obtained by clearing* Δ if $\mathcal{T}'$ is a tangle in Γ' of order $\theta - 4\lambda - 2$, and

- $\mathcal{T}'$ is respectful with a metric $m_{\mathcal{T}'}$, and
- $\mathcal{T}'$ is conformal with $\mathcal{T}$, and
- if x, y are atoms of Γ and x', y' are atoms of Γ' with $x \subseteq x'$ and $y \subseteq y'$, then $m_{\mathcal{T}}(x, y) \geq m_{\mathcal{T}'}(x', y') \geq m_{\mathcal{T}}(x, y) - 4\lambda - 2$.

Theorem 5.5 ([12, Theorem (7.10)]). *Let Δ be a λ-zone. If $\theta \geq 4\lambda + 3$, then there exists a unique respectful tangle of order $\theta - 4\lambda - 2$ obtained by clearing Δ.*

Theorem 5.6 ([14, Theorem (9.2)]). *Let Γ be a 2-cell drawing in a surface Σ, and let $\mathcal{T}$ be a respectful tangle in Γ of order θ. Let x be an atom of Γ, and λ an integer with $2 \leq \lambda \leq \theta - 4$. Then there exists a $(\lambda + 3)$-zone Δ around x such that $x' \subseteq \Delta$ for every atom x' of Γ with $m_{\mathcal{T}}(x, x') \leq \lambda$.*

Lemma 5.7. *Let Γ be a 2-cell drawing in a surface, z an atom, and $\mathcal{T}$ a respectful tangle in Γ of order θ. Let λ be a nonnegative integer, and let C be the cycle of the boundary of a λ-zone around z. If $\theta \geq \lambda + 8$, then there exists a $(\lambda + 7)$-zone Λ around z such that the cycle bounding Λ is disjoint from C.*

Proof. For every atom x of Γ, let Λ_x be a 4-zone around x containing all atoms y with $m_{\mathcal{T}}(x, y) \leq 1$, and let Δ_x be the closure of Λ_x, and let C_x be the boundary cycle of Δ_x. For every $v \in V(C)$, since every region incident with v has distance 1 from v, v is an interior point of Δ_v. Let $\Delta = \Delta' \cup \bigcup_{v \in V(C)} \Delta_v$, where Δ' is the open disk with the boundary C. So $V(C)$ are interior points of Δ. By the triangle-inequality, for every $v \in V(C)$ and for every vertex u in Δ_v, $m_{\mathcal{T}}(z, u) \leq \lambda + 4$. Therefore, there exists a $(\lambda + 7)$-zone Λ around z that contains Δ by Theorem 5.6. Since any vertex in C is an interior point of Δ, it is an interior point of Λ, so C is disjoint from the cycle that bounds Λ. ∎

Let Σ be a connected surface, and let $\Delta_1, ..., \Delta_t$ be pairwise disjoint closed disks in Σ. Let Γ be a drawing in Σ such that $U(\Gamma) \cap \Delta_i = V(\Gamma) \cap \partial\Delta_i$ for

$1 \le i \le t$. Let $Z = \bigcup_{i=1}^{t} V(\Gamma) \cap \partial\Delta_i$. We say that a partition $(Z_1, Z_2, ..., Z_p)$ of Z satisfies the *topological feasibility condition* if there exist pairwise disjoint disks $D_1, D_2, ..., D_p$ in Σ such that $D_j \cap (\bigcup_{i=1}^{t} \Delta_i) = Z_j$ for $1 \le j \le p$.

Theorem 5.8 ([12, Theorem (3.2)])**.** *For every connected surface Σ and all integers $t \ge 0$ and $z \ge 0$, there exists a positive integer $\theta \ge z$ such that the following is true. Let $\Delta_1, ..., \Delta_t$ be pairwise disjoint closed disks in Σ, and let Γ be a 2-cell drawing in Σ such that $U(\Gamma) \cap \Delta_i = V(\Gamma) \cap \partial\Delta_i$ for $1 \le i \le t$. Let $|Z| \le z$, where $Z = \bigcup_{i=1}^{t} V(\Gamma) \cap \partial\Delta_i$, and let $(Z_1, Z_2, ..., Z_p)$ be a partition of Z satisfying the topological feasibility condition. Let $\mathcal{T}$ be a respectful tangle of order at least θ in Γ with metric $m_{\mathcal{T}}$ such that $m_{\mathcal{T}}(r_i, r_j) \ge \theta$ for $1 \le i < j \le t$, where r_i is the region of Γ meeting Δ_i for $1 \le i \le t$, and $V(\Gamma) \cap \partial\Delta_i$ is free for $1 \le i \le t$. Then there are mutually disjoint connected drawing $\Gamma_1, \Gamma_2, ..., \Gamma_p$ of Γ with $V(\Gamma_j) \cap Z = Z_j$ for $1 \le j \le p$.*

6 Excluding subdivision of a fixed graph

Let G be a graph and $\mathcal{T}$ a tangle in G. Given an integer k, a vertex v of G is said to be *k-free* (with respect to $\mathcal{T}$) if there is no $(A, B) \in \mathcal{T}$ of order less than k such that $v \in V(A) - V(B)$. Similarly, we say that a subgraph X of G is *k-free* (with respect to $\mathcal{T}$) if there is no $(A, B) \in \mathcal{T}$ of order less than k such that $V(X) \subseteq V(A) - V(B)$.

The *skeleton* of a proper arrangement α of a segregation $\mathcal{S}$ in Σ is the drawing $\Gamma = (U, V)$ in Σ with $V(\Gamma) = \bigcup_{v \in V(\mathcal{S})} \alpha(v)$ such that $U(\Gamma)$ consists of the boundary of $\alpha(S, \Omega)$ for each $(S, \Omega) \in \mathcal{S}$ with $|\bar{\Omega}| = 3$, and a line in $\alpha(S', \Omega')$ with ends $\overline{\Omega'}$ for each $(S', \Omega') \in \mathcal{S}$ with $|\overline{\Omega'}| = 2$. Note that we do not add any edges into the skeleton for (S, Ω) with $|\bar{\Omega}| \le 1$ or $|\bar{\Omega}| > 3$.

Lemma 6.1. *Let t, ρ, θ be nonnegative integers. Let G be a graph and $\mathcal{T}$ a tangle in G of order at least θ. Let α be a proper arrangement of a segregation $\mathcal{S}$ of G in a surface Σ. Let $(S, \Omega) \in \mathcal{S}$ be a ρ-vertex. Let G' be the skeleton of α and $\mathcal{T}'$ a respectful tangle in G' of order θ conformal with $\mathcal{T}$. If G' is 2-cell and $\theta \ge 2t + 9$, then there exists a cycle C such that the following hold.*

1. *C bounds a $(2t + 8)$-zone Λ in G' around some vertex in $\bar{\Omega}$.*

2. *Λ contains every vertex x of G' with $m_{\mathcal{T}'}(x, y) \le t$ for some $y \in \bar{\Omega}$.*

3. *The closure of Λ contains $\alpha(S, \Omega)$.*

4. *Let S' be the union of S'' over all societies $(S'', \Omega'') \in \mathcal{S}$ with $\alpha(S'', \Omega'')$ contained in the closure of Λ. Let Ω' be the cyclic ordering on $V(C)$ that coincides with the cyclic ordering of C. Then (S', Ω') is a $(\rho+4t+16)$-vortex.*

Proof. Let y be a vertex in $\bar{\Omega}$. By Theorem 5.6, there exists a $(t+5)$-zone Λ' around y in G' such that $x \in \Lambda'$ for every $x \in V(G')$ with $m_{\mathcal{T}'}(x, y) \leq t+2$. Since $m_{\mathcal{T}'}(y', y'') \leq 2$ for every two vertices y', y'' in $\bar{\Omega}$, $x \in \Lambda'$ for every $x \in V(G')$ with $m_{\mathcal{T}'}(x, z) \leq t$ for some $z \in \bar{\Omega}$. Let H be the drawing obtained from G' by deleting every vertex $x \in V(G')$ with $m_{\mathcal{T}'}(x, y) \leq t+2$. It follows from [11, Theorem (8.10)] that H has a region f homeomorphic to an open disk that contains $\alpha(S, \Omega)$ and all deleted vertices.
Claim 1: For every vertex v of H incident with f, there exists a closed walk ℓ_v of length at most $2t+8$ in the radial drawing of G' with $v, y \subseteq \mathrm{ins}(\ell_v)$ such that v is adjacent to only one vertex in ℓ_v and $V(\ell_v) \cap V(H) = \{v\}$.
Proof of Claim 1: Since v is incident with f, there exists a path P of length two in the radial drawing of G' containing v and a vertex v' of $G' - V(H)$ internally disjoint from $V(H)$. As $m_{\mathcal{T}'}(v', y) \leq t+2$, there exists a closed walk $W_{v'}$ of length at most $2t+4$ in the radial drawing of G' such that $\{v', y\} \subseteq \mathrm{ins}(W_{v'})$. Note that $v \in V(H)$, so $m_{\mathcal{T}'}(v, y) > t+2$ and $\{v\} \not\subseteq \mathrm{ins}(W_{v'})$. Hence, there exists a closed walk ℓ_v of length at most $2t+8$ in $W_{v'} \cup P$ with $\{v, y\} \subseteq \mathrm{ins}(\ell_v)$ and such that v is adjacent to only one vertex in ℓ_v. $\square$

We define $\mathcal{L}_v$ to be the set of all ℓ_v's mentioned in Claim 1 for each vertex v incident with f, and let $\mathcal{Q}_v$ be the set of closed walks W in the radial drawing with $V(W)$ contained in the union of any two members of $\mathcal{L}_v$. Note that if the boundary of f has a cut-vertex v, then there exists a block of H containing v and contained in $\mathrm{ins}(W)$ for some $W \in \mathcal{Q}_v$. Define L to be the graph obtained from H by deleting $\bigcup_v \bigcup_{W \in \mathcal{Q}_v} (\mathrm{ins}(W) - \{v\})$, where the first union runs through all vertices v incident with f. As $V(\ell_v) \cap V(H) = \{v\}$ for every vertex v incident with f and $\ell_v \in \mathcal{L}_v$, there exists a cycle C in L such that C is contained in the boundary of f. Observe that Claim 1 implies that $m_{\mathcal{T}'}(v, y) \leq t+4$ for every $v \in V(C)$. And every vertex in $V(H) - V(L)$ is in the inside of a closed walk of length at most $4t+16$ in the radial drawing of W', so C is the boundary of a $(2t+8)$-zone.

Let S' be the union of S'' over all societies $(S'', \Omega'') \in \mathcal{S}$ with $\alpha(S'', \Omega'')$ contained in the closure of the disk bounded by C. Let Ω' be the cyclic ordering on $V(C)$ that coincides with the cyclic ordering of C. Since (S, Ω)

is a ρ-vortex, for every two intervals I, J that partition $\bar{\Omega}$, there exists $X_{I,J} \subseteq V(S)$ with $|X_{I,J}| \leq \rho$ such that there exists no path in S from $I - X_{I,J}$ to $J - X_{I,J}$.

Now we prove that (S', Ω') is a $(\rho + 4t + 16)$-vortex. Let I', J' be two intervals that partition $\overline{\Omega'}$, let u, v be the first vertex in I', J', respectively, under the ordering Ω', and let $\ell_u^* \in \mathcal{L}_u$ and $\ell_v^* \in \mathcal{L}_v$. Let u' be a vertex in $V(\ell_u^*) \cap \bar{\Omega}$ and v' a vertex in $V(\ell_v^*) \cap \bar{\Omega}$, and let P_u be a path in ℓ_u^* from u to u' and P_v a path in ℓ_v^* from v to v'. If $V(P_u) \cap V(P_v) = \emptyset$, then let I'', J'' be the two intervals partitioning $\bar{\Omega}$ with the first vertex u', v', respectively. In this case, there does not exist a path from $I' - X'$ to $J' - X'$ in $S' - X'$, where $X' = V(P_u) \cup V(P_v) \cup X_{I'',J''}$ has size at most $\rho + 4t + 16$. If $V(P_u) \cap V(P_v) \neq \emptyset$, then there exists a path Q in $P_u \cup P_v$ from u to v such that there exists no path in $S' - V(Q)$ from $I' - V(Q)$ to $J' - V(Q)$. Note that $|V(Q)| \leq 4t + 16$. Therefore, (S', Ω') is a $(\rho + 4t + 16)$-vortex. ■

Lemma 6.2. *Let $d \geq 3$, and let $\kappa, h, h_1, h_2, ..., h_\kappa, \rho, \theta''$ be nonnegative integers. Then there exist integers $\theta_0(d, h, \rho, \kappa, \theta'')$, $\beta(d, h, \rho)$ and $f(d, h, \rho, \kappa)$ such that the following holds. Suppose that*

1. *G is a graph and $\mathcal{T}$ is a tangle in G, and*
2. *τ is a proper arrangement of a $\mathcal{T}$-central segregation $\mathcal{S}$ of G in a surface Σ, and*
3. *G' is the skeleton of τ, G' is 2-cell, and $\mathcal{T}'$ is a respectful tangle in G' of order θ, for some $\theta \geq \theta_0$, such that G contains G' as a minor and $\mathcal{T}'$ is conformal with $\mathcal{T}$, and*
4. *let $(S_1, \Omega_1), ..., (S_\kappa, \Omega_\kappa)$ be societies in $\mathcal{S}$, where each (S_i, Ω_i) is a ρ-vortex and contains at least one d-free vertex with respect to $\mathcal{T}$ such that for every $1 \leq i < j \leq \kappa$, and for every $x \in \overline{\Omega_i}$ and $y \in \overline{\Omega_j}$, $m_{\mathcal{T}'}(x, y) \geq 2f + 1$, and*
5. *$m_{\mathcal{T}'}(x, y) \geq f + 1$, for every $x \in \overline{\Omega_i}$ with $1 \leq i \leq \kappa$, and for every $y \in \bar{\Omega}$ with $(S, \Omega) \in \mathcal{S}$ and $|\bar{\Omega}| > 3$, and*
6. *$h_i \leq h$ for $1 \leq i \leq \kappa$.*

Then there exist $Z_1, Z_2, ..., Z_\kappa, U_1, U_2, ..., U_\kappa \subseteq V(G)$, a subdrawing $G'' = G' - \bigcup_{i=1}^{\kappa}(Z_i \cup U_i)$ of G', a tangle $\mathcal{T}''$ in G'' of order at least θ'' conformal with $\mathcal{T}'$ obtained from $\mathcal{T}' - \bigcup_{i=1}^{\kappa} Z_i$ by clearing at most κ f-zones in G' such that for every $i \in \{1, 2, ..., \kappa\}$, either

1. *$h_i \geq 2$, $U_i = \emptyset$ and $|Z_i| \leq \beta$ such that every vertex in $S_i - Z_i$ is not d-free with respect to $\mathcal{T}''$, or*

2. *$Z_i = \emptyset$, and U_i is the set of vertices of G contained in an f-zone Λ_i in G' around a vertex in $\overline{\Omega_i}$ with the boundary cycle Y_i, and there exist h_i subsets $A_{i,1}, A_{i,2}, ..., A_{i,h_i}$ of Y_i such that the following hold.*

   (a) $V(S_i) \subseteq U_i$.

   (b) Each $A_{i,j}$ has size d and $\bigcup_{j=1}^{h_i} A_{i,j}$ is free in G'' with respect to $\mathcal{T}''$.

   (c) $I_j \cap I_k = \emptyset$ for $1 \leq j < k \leq h_i$, where I_j, I_k is the minimum interval of Y_i containing $A_{i,j}, A_{i,k}$, respectively.

   (d) There exist $v_{i,1}, v_{i,2}, ..., v_{i,h_i} \in U_i$ such that there are h_i disjoint d-spiders in G contained in Λ_i, where the j-th spider is from $v_{i,j}$ to $A_{i,j}$.

Proof. Define k' to be the value $k'(h, d, \rho)$ mentioned in Theorem 4.2, and let $\beta(d, h, \rho) = 2(k'd)^{d+1} + 1$. Define $s' = s'_{4.2}(h, d, 4hd + 3, \rho) + 2hd + \kappa\beta$, where $s'_{4.2}$ is the value s' mentioned in Theorem 4.2. Let $f(d, h, \rho, \kappa) = 36 + 10s'$ and $\theta_0(d, h, \rho, \kappa) = \theta'' + \kappa(4f + \beta + 2)$. Let $i \in \{1, 2, ..., \kappa\}$ be fixed. For simplicity, we denote (S_i, Ω_i) by (S, Ω), and let v_S be a vertex in $\bar{\Omega}$. Let $\Lambda'_{S,0}$ be a 5-zone in G' around v_S such that $\Lambda'_{S,0}$ contains all atoms y of G' with $m_{\mathcal{T}'}(v_S, y) \leq 2$ in its interior. Note that every vertex in $\bar{\Omega}$ has distance at most 2 from v_S with respect to the metric $m_{\mathcal{T}'}$, so $\Lambda'_{S,0}$ contains $\tau(S, \Omega)$. Let $\Lambda_{S,0}$ be an 18-zone in G' around a vertex in $\bar{\Omega}$ such that $\Lambda_{S,0}$ satisfies Lemma 6.1 and contains every vertex of $\Lambda'_{S,0} \cap G$ and $\bar{\Omega}$ are interior points of $\Lambda_{S,0}$. Let $G_{S,0}$ be the union of S' over all societies (S', Ω') with $\tau(S', \Omega')$ contained in the closure of $\Lambda_{S,0}$, and let $C_{S,0}$ be the boundary cycle of $\Lambda_{S,0}$. Let $(G_{S,0}, \Omega_{S,0})$ be a society, where $\overline{\Omega_{S,0}} = V(C_{S,0})$ with the cyclic ordering determined by $C_{S,0}$. Note that Lemma 6.1 ensures that $(G_{S,0}, \Omega_{S,0})$ is a $(\rho + 36)$-vortex.

For $1 \leq j \leq s'$, let $\Lambda_{S,j}$ be a $(36 + 10j)$-zone around v_S such that $\Lambda_{S,j}$ contains every vertex x of G'' with $m_{\mathcal{T}'}(x, v_S) \leq 36 + 10(j - 1)$ and $\partial\Lambda_{S,j} \cap \partial\Lambda_{S,j-1} = \emptyset$. Note that the existence of $\Lambda_{S,j}$ follows from Lemmas 5.6 and 5.7. Let $C_{S,j}$ be the boundary cycle of $\Lambda_{S,j}$ for $1 \leq j \leq s'$. Let $\Lambda_S = \Lambda_{S,s'}$. Let G_S be the union of S' over all societies (S', Ω') with $\tau(S', \Omega')$ contained in the closure of Λ_S, and let Ω_S be the cyclic ordering on the boundary cycle of Λ_S. So (G_S, Ω_S) is a composition of a $(\rho+36)$-vortex $(G_{S,0}, \Omega_{S,0})$ with a rural neighborhood which has a presentation with an s'-nest $(C_{S,1}, C_{S,2}, ..., C_{S,s'})$.

Let $h'_i = k'$ if $h_i \neq 1$, and $h'_i = 1$ if $h_i = 1$. Let X_S be the set of d-free vertices in S with respect to $\mathcal{T}$. Note that $X_S \neq \emptyset$ by assumption. By Theorem 3.5, either there exist h'_i disjoint d-spiders from X_S to $\overline{\Omega_S}$, or there exists $W_S \subseteq V(G) \cap \Lambda_S$ with $|W_S| \leq 2(h'_i d)^{d+1} + 1 \leq \beta$ such that every d-spider from X_S to $\overline{\Omega_S}$ intersects W_S. When the latter case holds and $h_i > 1$, the first statement of the theorem holds by taking $U_i = \emptyset$ and $Z_i = W_S$. When $h_i = 1$, the former case holds by Menger's Theorem and the fact that S contains a d-free vertex. Therefore, we assume that the former case holds.

Define Z_i to be the empty set. Let $D_{i,1}, D_{i,2}, ..., D_{i,h'_i}$ be disjoint d-spiders from X_{S_i} to $\overline{\Omega_{S_i}}$. Apply Theorem 4.2 by taking $(S, \Omega) = (G_{S_i}, \Omega_{S_i})$, $(S_0, \Omega_0) = (S_i, \Omega_i)$ and $D_j = D_{i,j}$ for $1 \leq j \leq h'_i$, there exist pairwise disjoint d-spiders $D'_{i,1}, D'_{i,2}, ..., D'_{i,h_i}$ from X_{S_i} to $V(C_{S_i,s'})$, a $(4hd+3)$-nest $(N_{S_i,1}, ..., N_{S_i,4hd+3})$ and intervals $I_{i,1}, I_{i,2}, ..., I_{i,h_i}$ of $C_{S_i,s'}$ satisfying the conclusions of Theorem 4.2. For every $1 \leq j \leq h_i$, since each $D'_{i,j}$ is perpendicular to $(N_{S_i,1}, ..., N_{S_i,4hd+3})$, there exists a set $A_{i,j}$ of $h_i d$ vertices in $D'_{i,j} \cap V(N_{S_i,1})$ such that there exist $h_i d$ disjoint paths from $A_{i,j}$ to $V(C_{S_i,s'})$, but there exists no path from $D'_{i,j} \cap X_{S_i}$ to $V(N_{S_i,1})$ in $D'_{i,j} - A_{i,j}$. Note that $N_{S_i,1}$ is contained in the disk bounded by $C_{S_i,s'}$ which bounds an f-zone, so $N_{S_i,1}$ is the boundary of an f-zone. Define U_i to be the set of vertices of G inside the open disk bounded by $N_{S_i,1}$. Define $G'' = G' - \bigcup_{i=1}^{\kappa}(Z_i \cup U_i)$. So G'' is a subgraph of $G' - \bigcup_{j=1}^{\kappa} Z_i$ obtained by clearing at most κ f-zones. By Theorem 5.5, there exists a tangle $\mathcal{T}''$ of G'' of order $\theta - \kappa\beta - \kappa(4f+2) \geq \theta''$ obtained from $\mathcal{T}' - \bigcup_{i=1}^{\kappa} Z_i$ by clearing at most κ f-zones. Therefore, $\mathcal{T}''$ is conformal with $\mathcal{T}' - \bigcup_{i=1}^{\kappa} Z_i$. On the other hand, by planarity, for every $1 \leq j \leq h_i$, there exists an interval $J_{i,j}$ of $N_{S_i,1}$ containing $A_{i,j}$, such that $J_{i,j} \cap J_{i,j'} = \emptyset$ for every $j' \neq j$.

To prove this lemma, it is sufficient to show that $\bigcup_{j=1}^{h_i} A_{i,j}$ is free with respect to $\mathcal{T}''$. We first prove that $m_{\mathcal{T}''}(x,y) \geq 2h_i d + 1$ for every atom x in $A_{i,j}$ and atom $y \in \overline{\Omega_{S_i}}$. Suppose to the contrary that $m_{\mathcal{T}''}(x,y) \leq 2h_i d$ for some $x \in A_{i,j}$ and $y \in \overline{\Omega_{S_i}}$. So there exists a closed walk W of length at most $4h_i d$ in the radial drawing of G'' such that $\{x,y\} \subseteq \text{ins}(W)$. Since $(N_{S_i,2}, ..., N_{S_i,4hd+2})$ is a $(4hd+1)$-nest such that x is inside the open disk bounded by $N_{S_i,2}$ and y is outside the closed disk bounded by $N_{S_i,4hd+2}$, $\{y\}$ and the closed disk bounded by $N_{S_i,4hd+2}$ are contained in $\text{ins}(W)$. Note that $V(W) \cap U_{i'} = \emptyset$ for every $i' \neq i$, otherwise $V(W) \cap V(N_{S_{i'},j}) \neq \emptyset$ for $2 \leq j \leq 4hd+2$, since each $U_{i'}$ is contained in the disk bounded by $N_{S_{i'},1}$. Therefore, there exists a closed walk W' of length at most $|V(W)| +$

$\sum_{j=1}^{\kappa}|Z_j| \leq 4hd + 2\kappa\beta$ in the radial drawing of G' such that $\{y\}$ and the closed disk bounded by $N_{S_i,1}$ are contained in $\mathrm{ins}(W')$. In other words, $m_{\mathcal{T}'}(v_S, y) \leq 2hd + \kappa\beta$. But then $\Lambda_{i,s'-1}$ contains y, a contradiction. Hence, $m_{\mathcal{T}''}(x, y) \geq 2h_i d + 1$ for every atom x in $A_{i,j}$ and atom $y \in \overline{\Omega_{S_i}}$.

Suppose that $\bigcup_{j=1}^{h_i} A_{i,j}$ is not free with respect to $\mathcal{T}''$ for some i, then there exists $(A, B) \in \mathcal{T}''$ such that $\bigcup_{j=1}^{h_i} A_{i,j} \subseteq V(A)$ with order less than dh_i. We assume that A is as small as possible, so $m_{\mathcal{T}''}(x, y) < dh$ for every atom x in A and $y \in V(A) \cap V(B)$. That is, $m_{\mathcal{T}''}(x, y) < 2dh$ for every atoms x, y in A. Therefore, $\overline{\Omega_{S_i}} \subseteq V(B) - V(A)$. However, there exist dh_i disjoint paths from $\bigcup_{j=1}^{h_i} A_{i,j}$ to $\overline{\Omega_{S_i}}$, a contradiction. So $\bigcup_{j=1}^{h_i} A_{i,j}$ is free with respect to $\mathcal{T}''$ for every i. This proves the lemma. ■

Lemma 6.3. *Let $d \geq 3, h$ be positive integers. Let G be a 2-cell drawing in a surface Σ, and let $\mathcal{T}$ be a respectful tangle in G. Then there exist integers $\theta(d, h, \Sigma), \phi(d, h, \Sigma)$ such that if $\mathcal{T}$ has order at least θ and G contains h d-free vertices $v_1, v_2, ..., v_h$ with $m_{\mathcal{T}}(v_i, v_j) > \phi$ for $1 \leq i < j \leq h$, then G admits an H-subdivision for every graph H of order h and of maximum degree d embeddable in Σ.*

Proof. Let H be a graph of order h and of maximum degree d embeddable in Σ. Let $\theta_{5.8}$ be the positive integer θ mentioned in Theorem 5.8 by taking $t = h$ and $z = dh$. Note that $(\{v_i\}, \{v_i\})$ is a 0-vertex for every i. For $1 \leq i \leq h$, let Λ_i be the 12-zone around v_i of G mentioned in Lemma 6.1 such that Λ_i contains v_i and all its neighbors, and let S_i be the subgraph of G that is the union of S' over all societies (S', Ω') with $\alpha(S', \Omega')$ contained in the inside the closure of Λ_i, and let $\overline{\Omega_i} = \partial\Lambda_i \cap V(G)$ with the cyclic order defined by the boundary cycle of Λ_i. So (S_i, Ω_i) is a 24-vortex. Let $\theta' = \theta_{6.2}(d, 1, 24, h, \theta_{5.8})$, $\beta = \beta_{6.2}(d, 1, 24)$ and $f = f_{6.2}(d, 1, 24, \kappa)$, where $\theta_{6.2}$, $\beta_{6.2}$ and $f_{6.2}$ be the numbers θ_0, β, f mentioned in Lemma 6.2. Define $\theta = \theta' + h(4f + 2) + 2f + 1$ and $\phi = \theta_{5.8} + h(4f + 2) + 2f + 1$.

Applying Lemma 6.2 by taking $\kappa = h$, $h_i = 1$ for $1 \leq i \leq h$, $\rho = 24$, $\theta'' = \theta_{5.8}$, and $\mathcal{S}$ the segregation consisting of $(S_1, \Omega_1), (S_2, \Omega_2), ..., (S_h, \Omega_h)$ and the societies in which each of them consists of exactly one edge that is not in $\bigcup_{i=1}^{h} S_i$, we obtain the desired subgraph G'' with a respectful tangle $\mathcal{T}''$, and $A_{i,1}$ for $1 \leq i \leq h$, such that every $A_{i,1}$ is free with respect to $\mathcal{T}''$, as mentioned in the conclusion of Lemma 6.2. Then for every $x \in A_{i,1}$ and $y \in A_{j,1}$ for some $i \neq j$, we have that $m_{\mathcal{T}''}(x, y) \geq \theta_{5.8}$ by Theorem 5.5.

For $1 \leq i \leq h$, let Δ_i be a closed disk in Σ contained in the closure of Λ_i such that $\Delta_i \cap G'' = A_{i,1}$. Since H can be embedded in Σ, we can partition $\bigcup_{i=1}^{h} A_{i,1}$ and apply Theorem 5.8 to obtain a linear forest so that an H-subdivision in G can be obtained by concatenating these linear forests and h disjoint d-spiders in $S_1, S_2, ..., S_h$, where each S_i is from v_i to $A_{i,1}$, we obtain an H-subdivision in G. ■

Lemma 6.4. *Let ρ be an integer, G a graph, $\mathcal{T}$ a tangle in G of order at least $2\rho + 2$, and $\mathcal{S}$ a segregation of G. If $(S, \Omega) \in \mathcal{S}$ is a ρ-vortex and there exists no $(A, B) \in \mathcal{T}$ of order at most $2\rho + 1$ such that $B \subseteq S$, then there exists no $(A', B') \in \mathcal{T}$ of order at most the half of the order of $\mathcal{T}$ such that $B' \subseteq S$.*

Proof. Suppose that there exists $(A, B) \in \mathcal{T}$ of order at most the half of the order of $\mathcal{T}$ such that $B \subseteq S$. Let $\bar{\Omega} = v_1, v_2, ..., v_n$ in order, where $n = |\bar{\Omega}|$. We may assume that every v_i is adjacent to a vertex in $G - V(S)$, otherwise we may remove it from $\bar{\Omega}$. As (S, Ω) is a ρ-vortex, by Theorem 8.1 in [9], there exists a path decomposition $(P, \mathcal{X})$ of S of adhesion at most ρ such that the i-th bag X_i of $(P, \mathcal{X})$ contains v_i for every $1 \leq i \leq n$. For every subgraph H of S, we define (A_H, B_H) to be the separation of G with minimum order such that $A_H = H$. In particular, for $1 \leq i \leq n$, $(A_{S[X_i]}, B_{S[X_i]})$ has order at most $2\rho+1$, so $(A_{S[X_i]}, B_{S[X_i]}) \in \mathcal{T}$. For $1 \leq i \leq n$, define $(A_i, B_i) = (A \cup A_{S[\bigcup_{j=1}^{i} X_j]}, B \cap B_{S[\bigcup_{j=1}^{i} X_j]})$. Note that if $v_i \in V(B)$, then $v_i \in V(A)$ since $B \subseteq S$ and v_i is adjacent to a vertex in $G - V(S)$. So the order of (A_i, B_i) is at most $|V(A) \cap V(B)| + |V(A_{S[\bigcup_{j=1}^{i} X_j]}) \cap V(B_{S[\bigcup_{j=1}^{i} X_j]}) \cap (V(B) - V(A))| \leq |V(A) \cap V(B)| + \rho$. Since the order of (A, B) is at most the half of the order of $\mathcal{T}$, and the order of $\mathcal{T}$ is greater than 2ρ, either $(A_i, B_i) \in \mathcal{T}$ or $(B_i, A_i) \in \mathcal{T}$ by the first tangle axiom. Let $(A_0, B_0) = (A, B)$. We shall prove that $(A_i, B_i) \in \mathcal{T}$ for $0 \leq i \leq n$ by induction on i.

When $i = 0$, $(A_0, B_0) = (A, B) \in \mathcal{T}$. Assume that $(A_i, B_i) \in \mathcal{T}$ for some i. Suppose that $(B_{i+1}, A_{i+1}) \in \mathcal{T}$. But $(A_i, B_i), (A_{S[X_{i+1}]}, B_{S[X_{i+1}]}) \in \mathcal{T}$, and $B_{i+1} \cup A_i \cup S[X_{i+1}] = G$, a contradiction. This proves that $(A_i, B_i) \in \mathcal{T}$ for every $0 \leq i \leq n$.

Furthermore, $(A_n, B_n) = (A \cup S, B \cap B_S)$. Recall that $V(B \cap B_S) \subseteq V(B) \cap \bar{\Omega} \subseteq V(A) \cap V(B)$, so $|V(B_n)| \leq |V(A) \cap V(B)|$. Hence, $(B_n, G - E(B_n))$ has order less the order of $\mathcal{T}$, so $(B_n, G - E(B_n)) \in \mathcal{T}$ by the third tangle axiom. However, $A_n \cup B_n = G$, contradicting the second axiom. This completes the proof. ■

Given a proper arrangement α of a segregation $\mathcal{S}$ in a surface Σ, we say that the *trunk* of α is the drawing $\Gamma = (U, V)$ in Σ, where $V(\Gamma) = \bigcup_{v \in V(\mathcal{S})} \alpha(v)$, and $U(\Gamma)$ consists of the following.

- The boundary of $\alpha(S, \Omega)$ for each $(S, \Omega) \in \mathcal{S}$ with $|\bar{\Omega}| \geq 3$.
- The boundary of $\alpha(S, \Omega)$ for each $(S, \Omega) \in \mathcal{S}$ with $|\bar{\Omega}| = 2$ such that there exist two edge-disjoint paths in S connecting the two vertices in $\bar{\Omega}$.
- A line in $\alpha(S, \Omega)$ with ends $\bar{\Omega}$ for each $(S, \Omega) \in \mathcal{S}$ with $|\bar{\Omega}| = 2$ such that there do not exist two edge-disjoint paths in S connecting the two vertices in $\bar{\Omega}$.

Note that we do not add any edges into the trunk for (S, Ω) with $|\bar{\Omega}| \leq 1$.

The notion of trunk is very similar with the skeleton, and we will prove the following general lemma for skeletons and trunks. The notion of trunk will be used in a subsequent paper but not in the rest of this paper. We say a graph is *weakly subcubic* if every vertex is adjacent to at most three neighbors.

Lemma 6.5. *For a positive nondecreasing function ϕ, integers $\rho, \lambda, \kappa, k, \theta^*, d, s$ with $d \geq 4$, and every collection of graphs $\mathcal{F}$ on at most s vertices, there exist integers θ, ρ^* such that the following is true. Assume that a graph G has a tangle $\mathcal{T}$ and a $\mathcal{T}$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ that has a proper arrangement τ in a surface Σ such that the following hold.*

1. *$|\bar{\Omega}| \leq 3$ for every $(S, \Omega) \in \mathcal{S}_1$.*
2. *$|\mathcal{S}_2| \leq \kappa$.*
3. *(S, Ω) is a ρ-vortex for every $(S, \Omega) \in \mathcal{S}_2$.*
4. *Let G' be the skeleton of $\mathcal{S}$ or the trunk of $\mathcal{S}$. G' is 2-cell embedded in Σ and has a respectful tangle $\mathcal{T}'$ of order at least θ conformal with $\mathcal{T}$.*
5. *There exist k λ-zones $\Lambda_1, \Lambda_2, ..., \Lambda_k$ in G' with respect to the metric $m_{\mathcal{T}'}$ such that every d-free subgraph of G' with respect to $\mathcal{T}'$ isomorphic to a member of $\mathcal{F}$ is contained in $\bigcup_{i=1}^{k} \Lambda_i$.*
6. *If G' is the trunk of $\mathcal{S}$, then the following hold.*
   (a) *G' is weakly subcubic.*

(b) $S \cap S' = \emptyset$ *for different members* $(S,\Omega),(S',\Omega') \in \mathcal{S}_1$ *with* $|\bar{\Omega}| = |\overline{\Omega'}| = 3$.

(c) *For every* $(S,\Omega) \in \mathcal{S}_2$*, there exists a cycle in* S *passing through all vertices in* $\bar{\Omega}$ *in order.*

(d) *For every edge in a graph in* $\mathcal{F}$*, there exists another edge that has the same ends.*

Then there exists a $\mathcal{T}$*-central segregation* $\mathcal{S}^* = \mathcal{S}_1^* \cup \mathcal{S}_2^*$ *properly arranged in* Σ *such that the following hold.*

1. $\mathcal{S}_1^* \subseteq \mathcal{S}_1$*; in particular,* $|\bar{\Omega}| \leq 3$ *for every* $(S,\Omega) \in \mathcal{S}_1^*$.

2. $|\mathcal{S}_2^*| \leq \kappa + k$ *and* $\bigcup_{(S,\Omega)\in\mathcal{S}_2} S \subseteq \bigcup_{(S,\Omega)\in\mathcal{S}_2^*} S$.

3. *There exists an integer* ρ' *with* $\rho' \leq \rho^*$ *such that* (S,Ω) *is a* ρ'*-vortex for every* $(S,\Omega) \in \mathcal{S}_2^*$.

4. *Let* G^* *be the skeleton of* $\mathcal{S}^*$ *or the trunk of* $\mathcal{S}^*$*, respectively, if* G' *is the skeleton of* $\mathcal{S}$ *or the trunk of* $\mathcal{S}$*, respectively.* G^* *is* 2*-cell embedded in* Σ *and has a respectful tangle* $\mathcal{T}^*$ *of order at least* $\theta^* + \phi(\rho^*) + 2\rho^*$ *conformal with* $\mathcal{T}$.

5. *If for every* $(S',\Omega') \in \mathcal{S}_1$ *and for every* $x \in \overline{\Omega'}$*, there exist* $|\overline{\Omega'}| - 1$ *paths in* S' *from* x *to* $\overline{\Omega'} - \{x\}$ *intersecting only in* $\{x\}$*, then for every* $(S,\Omega) \in \mathcal{S}_2^*$*, there exists a cycle passing through all vertices in* $\overline{\Omega^*}$ *in order.*

6. *If* G^* *is the trunk of* $\mathcal{S}^*$*, then it is weakly subcubic.*

7. *There is no d-free subgraph of* G^* *with respect to* $\mathcal{T}^*$ *isomorphic to a member of* $\mathcal{F}$.

8. $m_{\mathcal{T}^*}(x,y) \geq \phi(\rho')$ *for every atoms* x,y *of* G^* *with* $x \in S_x, y \in S_y$ *for different members* $(S_x,\Omega_x),(S_y,\Omega_y) \in \mathcal{S}_2^*$,

Proof. Note that each society that consists of a single vertex is a 0-vortex. So by Lemma 6.1, for each Λ_i, we can find a $(2\lambda+8)$-zone Λ_i' containing Λ_i such that $(G \cap \Lambda_i, \Omega)$ is a $(4\lambda+16)$-vortex, where Ω is a cyclic ordering on $V(G) \cap \partial\Lambda_i$ consistent with the cyclic ordering of the cycle bounding Λ_i. Therefore, we can replace Λ_i by Λ_i' so that we may assume that every Λ_i is a λ'-zone and the subgraph of G inside the disk Λ_i is a λ'-vortex (S, Ω) for some constant λ' only depending on λ. Similarly, for each $(S, \Omega) \in \mathcal{S}_2$, there exists a 12-zone Λ_S containing the disk $\tau(S, \Omega)$, and the subgraph of G inside this disk is a $(\rho + 24)$-vortex by Lemma 6.1.

Let $\mathcal{C} = \{\Lambda_i, \Lambda_S : 1 \leq i \leq k, (S, \Omega) \in \mathcal{S}_2\}$, and let λ'' be the minimum t such that every member of $\mathcal{C}$ is a t-zone. For each member Λ of $\mathcal{C}$, let $S_\Lambda = G \cap \Lambda$, and let $\overline{\Omega_\Lambda} = V(G) \cap \partial\Lambda$ ordered by the cyclic ordering given by the cycle bounding Λ. Let M be the maximum depth of $(S_\Lambda, \Omega_\Lambda)$ for all members Λ of $\mathcal{C}$. Note that $|\mathcal{C}| \leq k + \kappa$, $M = \max\{\lambda', \rho + 24\}$, and $\lambda'' \leq \max\{\lambda', 12\}$. Then we consecutively test whether there exist two atoms of G' in different members of $\mathcal{C}$ with distance less than $\phi(M + 2) + (4\lambda'' + 2)|\mathcal{C}| + 4$ under the metric $m_{\mathcal{T}'}$, and if such two nearby vortices exist, then we do the following. Find a minimum number t such that the $(2t + 8)$-zone Λ mentioned in the conclusion of Lemma 6.1 contains these two nearby members of $\mathcal{C}$, and remove these two members from $\mathcal{C}$ and add Λ into $\mathcal{C}$, and then we update M and λ''. Since $|\mathcal{C}|$ decreases in each step, this process will terminate within $\kappa + k$ steps. Furthermore, when the process terminates, each member of $\mathcal{C}$ defines a M-vortex, where M only depends on ϕ, κ, k, λ and ρ, and the distance between two members of $\mathcal{C}$ is at least $\phi(M + 2) + (4\lambda'' + 2)|\mathcal{C}| + 4$ under the metric $m_{\mathcal{T}'}$. Clearly, there exists an integer ρ^* (that only depends on $\phi, \kappa, k, \lambda, \rho$) such that $M + 2 \leq \rho^*$. We define $\theta = 2\rho^*(\theta^* + \phi(\rho^*) + 2\rho^*) + 4\lambda'' + 16$.

For every $\Lambda \in \mathcal{C}$, let Λ' be the minimal closed disk in Σ containing Λ and $\tau(S, \Omega)$ for every $(S, \Omega) \in \mathcal{S}_1$ with $|\bar{\Omega} \cap \Lambda| \geq 2$. Clearly, Λ' is a $(\lambda'' + 2)$-zone, and $(S_{\Lambda'}, \Omega_{\Lambda'})$ is a $(M + 2)$-vortex, and every two atoms of G' in different members of $\mathcal{C}$ have distance at least $\phi(M + 2) + (4\lambda'' + 2)|\mathcal{C}|$. If G' is the skeleton of $\mathcal{S}$, then define $(S'_\Lambda, \Omega'_\Lambda)$ to be $(S_\Lambda, \Omega_\Lambda)$ for every $\Lambda \in \mathcal{C}$. Now assume that G' is the trunk of $\mathcal{S}$. Recall that G' is weakly subcubic and $S \cap S' = \emptyset$ for different members $(S, \Omega), (S', \Omega') \in \mathcal{S}_1$ with $|\bar{\Omega}| = |\overline{\Omega'}| = 3$ in this case. Observe that there is no $(S, \Omega) \in \mathcal{S}_1$ with $S \not\subseteq G \cap \Lambda'$ and $|\bar{\Omega} \cap \Lambda'| \geq 2$ unless $|\bar{\Omega}| \leq 2$, since $S \cap S' = \emptyset$ for different members $(S, \Omega), (S', \Omega') \in \mathcal{S}_1$ with $|\bar{\Omega}| = |\overline{\Omega'}| = 3$. We replace Λ' by the minimal disk that contains Λ' and $\tau(S, \Omega)$ for every $(S, \Omega) \in \mathcal{S}_1$ with $|\bar{\Omega}| \leq 2$. Then there is no $(S, \Omega) \in \mathcal{S}_1$ with $S \not\subseteq G \cap \Lambda'$ and $|\bar{\Omega} \cap \Lambda'| \geq 2$. Then we define $(S'_\Lambda, \Omega'_\Lambda)$ to be $(S_{\Lambda'}, \Omega_{\Lambda'})$ for every $\Lambda \in \mathcal{C}$. Note that in the both cases, if $\mathcal{S}$ satisfies the property that for every $(S, \Omega) \in \mathcal{S}_1$ and for every $x \in \bar{\Omega}$, there exist $|\bar{\Omega}| - 1$ paths in S from x to $\bar{\Omega} - \{x\}$, then there exists a cycle in S'_Λ passing through all vertices in $\overline{\Omega'_\Lambda}$ in order, since Λ' is bounded by a cycle in G'.

Define a new segregation $\mathcal{S}^* = \mathcal{S}^*_1 \cup \mathcal{S}^*_2$ of G by letting $\mathcal{S}^*_2 = \{(S'_\Lambda, \Omega'_\Lambda) :$

$\Lambda \in \mathcal{C}\}$ and $\mathcal{S}_1^* = \{(S,\Omega) \in \mathcal{S}_1 : V(S) \not\subseteq \bigcup_{\Lambda \in \mathcal{C}} V(S'_\Lambda)\}$. Let G^* be the skeleton (or trunk, respectively) of $\mathcal{S}^*$ if G' is the skeleton (or trunk, respectively) of $\mathcal{S}$. Observe that for every integer t and separation (A,B) of G' or G^* of order t, there exists a separation (A',B') of G of order at most $2\rho^* t$ such that $A \subseteq A'$ and $B \subseteq B'$, since every member of $\mathcal{S}_2$ or $\mathcal{S}_2^*$ has depth at most ρ^*. Similarly, for every G^*-normal O-arc in Σ that intersects in G^* at most t vertices, there exists a G'-normal O-arc in Σ that intersects in G' at most $2\rho^* t$ vertices. Therefore, there exists a tangle $\mathcal{T}^*$ in G^* of order at least $\theta/(2\rho^*) \geq \theta^* + \phi(\rho^*) + 2\rho^*$ conformal with $\mathcal{T}$ and $\mathcal{T}'$, and $\mathcal{T}^*$ is respectful. On the other hand, $\mathcal{T}^*$ can be obtained from $\mathcal{T}'$ by clearing at most $|\mathcal{C}|$ $(\lambda''+2)$-zones, so $m_{\mathcal{T}^*}(x,y) \geq m_{\mathcal{T}'}(x,y) - |\mathcal{C}|(4\lambda''+2) \geq \phi(M+2)$ by Theorem 5.5. Therefore, Conclusions 1-4 and 8 hold.

Recall that every member in $\mathcal{S}_2^*$ is a society obtained by applying Lemma 6.1, so Conclusion 5 holds. This implies that G' contains G^* as a subdivision. So if G' is the trunk of $\mathcal{S}$, then G^* is weakly subcubic as G' is. This proves Conclusion 6. In fact, G^* is a subgraph of G' if G' is the skeleton of $\mathcal{S}$. So Conclusion 7 holds in this case. But when G^* is the trunk of $\mathcal{S}^*$, there do not exist vertices x,y of G^* such that there are multiple edges between x,y in G^* but not in G'; otherwise, there exists a society $(S,\Omega) \in \mathcal{S}_1^*$ such that $S \not\subseteq G \cap \Lambda'$ and $|\bar{\Omega} \cap \Lambda'| \geq 2$, where Λ' is the λ''-zone corresponding to the vortex containing x,y, a contradiction. But when G^* is the trunk of $\mathcal{S}^*$, for every edge in a graph in $\mathcal{F}$, there exists another edge with the same ends, so no subgraph of G^* that is not a subgraph of G' but is isomorphic to a graph in $\mathcal{F}$. Hence, Conclusion 7 holds.

It remains to prove that $\mathcal{S}^*$ is a $\mathcal{T}$-central segregation of G. Since $\mathcal{T}'$ has order at least θ and is conformal with $\mathcal{T}$, the order of $\mathcal{T}$ is at least θ. Since $\mathcal{S}_1^* \subseteq \mathcal{S}_1$ and $\mathcal{S}$ is $\mathcal{T}$-central, by Lemma 6.4, it is sufficient to show that there is no $(A,B) \in \mathcal{T}$ of order at most $2\rho^*+1$ such that $B \subseteq S$ for some $(S,\Omega) \in \mathcal{S}_2^*$. Suppose that such (A,B) exists. Let (A',B') be a separation of G^* such that $V(A') = V(A) \cap V(G^*)$ and $V(B') = V(B) \cap V(G^*)$. Note that $(A',B') \in T^*$ since $\mathcal{T}^*$ is a tangle of order at least $2\rho^*+1$ conformal with $\mathcal{T}$. Since $B \subseteq S$, $V(B') \subseteq V(A) \cap V(B)$, so B' contains at most $2\rho^*+1$ vertices. However, the first and the second tangle axioms imply that $(G^* - E(B'), B') \in \mathcal{T}^*$, contradicting the third tangle axiom. Hence $\mathcal{S}^*$ is $\mathcal{T}$-central. ∎

A segregation $\mathcal{S}$ of G is *maximal* if there exists no segregation $\mathcal{S}'$ such that $\{(S,\Omega) \in \mathcal{S} : |\bar{\Omega}| > 3\} = \{(S',\Omega') \in \mathcal{S}' : |\overline{\Omega'}| > 3\}$ and for every

$(S,\Omega) \in \mathcal{S}$ with $|\bar{\Omega}| \leq 3$, there exists $(S',\Omega') \in \mathcal{S}'$ with $|\overline{\Omega'}| \leq 3$ such that $S' \subseteq S$, and the containment is strict for at least one society. Furthermore, if H is a triangle-free graph and the skeleton of a maximal segregation $\mathcal{S}$ of G admits an H-subdivision, then G admits an H-subdivision. Note that if a segregation $\mathcal{S}$ of G is maximal, then G contains the skeleton of $\mathcal{S}$ as a minor.

The following theorem is a stronger form of the structure theorem for excluding minors in [15].

Theorem 6.6 ([1, Theorem 7])**.** *For every graph L, there exists an integer κ such that for any nondecreasing positive function ϕ, there exist integers θ, ξ, ρ with the following property. Let $\mathcal{T}$ be a tangle of order at least θ in a graph G controlling no L-minor of G. Then there exist $Z \subseteq V(G)$ with size at most ξ and a maximal $(\mathcal{T} - Z)$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G - Z$ properly arranged in a surface Σ in which L cannot be drawn, where every $(S,\Omega) \in \mathcal{S}_1$ has the property that $|\bar{\Omega}| \leq 3$, and $|\mathcal{S}_2| \leq \kappa$ and every member in $\mathcal{S}_2$ is a p-vortex for some $p \leq \rho$. Furthermore, the skeleton G' of $\mathcal{S}$ is 2-cell embedded in Σ with a respectful tangle $\mathcal{T}'$ of order at least $\phi(p)$ conformal with $\mathcal{T} - Z$, and if x and y are two vertices in G' incident with two different members in $\mathcal{S}_2$, then $m_{\mathcal{T}'}(x,y) \geq \phi(p)$.*

Let us recall that the function mf was defined prior to Theorem 1.3. A graph H has a *nice* embedding in Σ if H can be 2-cell embedded in Σ and it has a set F of regions such that every vertex of H of degree at least 4 is incident with exactly one region in F, and $|F| = \mathrm{mf}(H,\Sigma)$.

Lemma 6.7 ([1, Lemma 12])**.** *Let H be a graph of maximum degree d that can be embedded in a surface Σ. Then there exists a triangle-free graph H' of maximum degree d admitting an H-subdivision such that $\mathrm{mf}(H',\Sigma) = \mathrm{mf}(H,\Sigma)$ and H' has a nice embedding in Σ.*

Recall that a vertex v in a graph G is d-free with respect to a tangle $\mathcal{T}$ in G if there does not exist a separation $(A,B) \in \mathcal{T}$ of order less than d such that $v \in V(A) - V(B)$. Now, we are ready to prove our main theorem, which we restate.

Theorem 6.8. *Let $d \geq 4, h$ be positive integers. Then there exist $\theta, \kappa, \rho, \xi, g \geq 0$ satisfying the following property. If H is a graph of maximum degree d on h vertices, and a graph G does not admit an H-subdivision, then for every tangle $\mathcal{T}$ in G of order at least θ, there exists $Z \subseteq V(G)$ with $|Z| \leq \xi$ such that either*

1. *no vertex of $G - Z$ is d-free with respect to $\mathcal{T} - Z$, or*
2. *there exist a $(\mathcal{T} - Z)$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G - Z$ with $|\mathcal{S}_2| \leq \kappa$, having a proper arrangement in some surface Σ of genus at most g such that every society (S_1, Ω_1) in $\mathcal{S}_1$ satisfies that $|\overline{\Omega_1}| \leq 3$, and every society (S_2, Ω_2) in $\mathcal{S}_2$ is a ρ-vortex, and satisfies the following property: either*
   - *(a) H cannot be drawn in Σ, or*
   - *(b) H can be drawn in Σ and $\mathrm{mf}(H, \Sigma) \geq 2$, and there exists $\mathcal{S}_2' \subseteq \mathcal{S}_2$ with $|\mathcal{S}_2'| \leq \mathrm{mf}(H, \Sigma) - 1$ such that every d-free vertex of $G - Z$ with respect to $\mathcal{T} - Z$ is in $S - \bar{\Omega}$ for some $(S, \Omega) \in \mathcal{S}_2'$.*

Proof. Note that there are only finitely many graphs of maximum degree d on h vertices, and there are only finitely many surfaces in which H can be drawn but $K_{\lceil \frac{3}{2}dh \rceil}$ cannot. So there exists h^* such that for every graph H on h vertices of maximum degree d and surface in which H can be drawn but $K_{\lceil \frac{3}{2}dh \rceil}$ cannot, the graph H' mentioned in Lemma 6.7 can be chosen such that it has at most h^* vertices.

We define the following.

- Let $\kappa_{6.6}$ be the number κ mentioned in Theorem 6.6 by taking $L = K_{\lceil \frac{3}{2}dh \rceil}$.
- Let $\theta_{6.2}, \beta_{6.2}, f_{6.2}$ be the functions θ_0, β, f mentioned in Lemma 6.2, respectively.
- Let ϕ' be the maximum $\phi_{6.3}(d, h^*, \Sigma)$ among all surfaces Σ in which $K_{\lceil \frac{3}{2}dh \rceil}$ cannot be drawn, where $\phi_{6.3}$ is the number ϕ mentioned in Lemma 6.3.
- Let $\theta_{5.8}$ be the maximum of θ mentioned in Theorem 5.8 by taking all surfaces in which $K_{\lceil \frac{3}{2}dh \rceil}$ cannot be drawn, and $t = h^*$, $z = dh^*$.
- Let $\phi^*(x) = 2f_{6.2}(d, h^*, x, \kappa_{6.6}) + \kappa_{6.6}(5\beta_{6.2}(d, h^*, x) + 2) + 2(dh^* + h^* + 1)(\theta_{5.8} + 6)$.
- Let $\theta'_{6.5}(x)$ be the function θ obtained by applying Lemma 6.5 by taking $\phi = \phi^*$, $\rho = x$, $\lambda = d + \phi' + 11$, $\kappa = \kappa_{6.6}$, $k = h^* + \kappa_{6.6}$, $\theta^* = \theta_{6.2}(d, h^*, x, \kappa_{6.6}, (dh^* + h^* + 1)(\theta_{5.8} + 1))$, $d = d$, $s = 1$ and $\mathcal{F}$ the set consisting of the graph that has exactly one vertex with no edges.

- Let $\theta_{6.6}$, $\xi_{6.6}$, $\rho_{6.6}$ be the number θ, ξ, ρ mentioned in Theorem 6.6, respectively, by taking $\kappa = \kappa_{6.6}$ and further taking $\phi(x) = \theta'_{6.5}(x)$.
- Let $\theta_{6.3}$ be the maximum of $\theta(d, h^*, \Sigma)$ mentioned in Lemma 6.3 among all surfaces Σ in which $K_{\lceil \frac{3}{2}dh \rceil}$ cannot be drawn.
- Let $\theta_{6.5}$ and $\rho_{6.5}$ be the numbers θ and ρ^* obtained by applying Lemma 6.5 by taking ϕ to be the function such that $\phi(x) = \phi^*(x)$, $\rho = \rho_{6.6}$, $\lambda = d + \phi' + 11$, $\kappa = \kappa_{6.6}$, $k = h^* + \kappa_{6.6}$, $\theta^* = \theta_{6.2}(d, h^*, \rho_{6.6}, \kappa_{6.6}, (dh^* + h^* + 1)(\theta_{5.8} + 1))$, $d = d$, $s = 1$ and $\mathcal{F}$ be the set consisting of the graph that has exactly one vertex with no edges.
- Let $\theta_{3.4} = (hd)^{d+1} + d$.

Now we are ready to define the numbers for the conclusion of this theorem.

- Let $\xi = \max\{\xi_{6.6} + (\kappa_{6.6} + h^*)\beta_{6.2}(d, h^*, \rho_{6.5}), (hd)^{d+1}\}$.
- Let $\theta = 2\rho_{\kappa_{6.6}+h^*}(\theta_{6.5} + \theta_{6.3} + \theta_{6.6}) + \xi$.
- Let $\kappa = \kappa_{6.6} + h^*$.
- Let $\rho = \rho_{\kappa_{6.6}} + h^*$.
- Let g be the maximum genus of a surface in which $K_{\lceil \frac{3}{2}dh \rceil}$ cannot be drawn.

Let $\mathcal{T}$ be a tangle of order at least θ in G. We may assume that G contains at least h vertices of degree at least d, otherwise the first statement holds by letting Z be the set of vertices of degree at least d. We first assume that $\mathcal{T}$ controls a $K_{\lceil \frac{3}{2}dh \rceil}$-minor. By Lemma 3.2 and Theorem 3.4, since G does not admit an H-subdivision, there exists a set of vertices Z of G with $|Z| \leq \xi$ such that for every vertex v of $G - Z$ of degree at least d in G, there exists a separation $(A_v, B_v) \in \mathcal{T} - Z$ of $G - Z$ of order at most $d - 1$ such that $v \in V(A_v) - V(B_v)$. Therefore, the first statement holds. So we may assume that $\mathcal{T}$ does not control a $K_{\lceil \frac{3}{2}dh \rceil}$-minor.

By Theorem 6.6, there exist a surface Σ in which $K_{\lceil \frac{3}{2}dh \rceil}$ cannot be drawn, $Z \subseteq V(G)$ with $|Z| \leq \xi_{6.6}$, and a maximal $(\mathcal{T} - Z)$-central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G - Z$ with $|\mathcal{S}_2| \leq \kappa_{6.6}$, having a proper arrangement τ in Σ such that every society (S, Ω) in $\mathcal{S}_1$ satisfies that $|\bar{\Omega}| \leq 3$, and every society in $\mathcal{S}_2$ is a $\rho_{6.6}$-vortex, and the skeleton G' of $\mathcal{S}$ is 2-cell embedded in Σ and has a respectful tangle $\mathcal{T}'$ of order at least $\phi(\rho_{6.6})$ conformal with $\mathcal{T}-Z$, and if x,y are two vertices in G' incident with two different members in $\mathcal{S}_2$, then $m_{\mathcal{T}'}(x,y) \geq \phi(\rho_{6.6})$. If H cannot be drawn in Σ, then Statement 2(a) holds, so we may assume that H can be drawn in Σ.

On the other hand, we may assume that $G-Z$ contains d-free vertices with respect to $\mathcal{T}-Z$, for otherwise Statement 1 holds. Note that every vertex in $\bigcup_{(S,\Omega)\in\mathcal{S}_1} V(S)-V(G')$ is not d-free with respect to $\mathcal{T}-Z$ since $d \geq 4$. If v is in $V((G-Z)\cap G')$ but is not d-free respect to $\mathcal{T}'$, then there exists a separation $(A',B') \in \mathcal{T}'$ of order less than d such that $v \in V(A')-V(B')$. We choose (A',B') such that A' is as small as possible. Note $m_{\mathcal{T}'}(v,x) < d$ for every $x \in V(A')$ by Theorem 5.3. Suppose that there is no vertex $x \in V(S)$ with $(S,\Omega) \in \mathcal{S}_2$ and $m_{\mathcal{T}'}(v,x) < d$. Then there exists $(A,B) \in \mathcal{T}-Z$ of order less than d such that $V(A') = \bigcup_{(S,\Omega)\in\mathcal{S}, V(S)\subseteq V(A)} \bar{\Omega}$ and $V(A)\cap V(B) = V(A')\cap V(B')$. So v is not d-free with respect to $\mathcal{T}-Z$. Therefore, if v is a vertex in $(G-Z)\cap G'$ that is d-free with respect to $\mathcal{T}-Z$ but not d-free with respect to $\mathcal{T}'$, then $m_{\mathcal{T}'}(v,x) < d$ for some $x \in V(S)$ with $(S,\Omega) \in \mathcal{S}_2$. By Theorem 5.6 and Lemma 5.7, for every $(S,\Omega) \in \mathcal{S}_2$, there exists a $(d+11)$-zone Λ_S with respect to $\mathcal{T}'$ around a vertex in $\bar{\Omega}$ containing every atom y with $m_{\mathcal{T}'}(x,y) \leq d+1$ as an interior point for all such x. Thus every vertex of $(G-Z)\cap G'$ that is d-free with respect to $\mathcal{T}-Z$ but not d-free with respect to $\mathcal{T}'$ is in $\bigcup_{(S,\Omega)\in\mathcal{S}_2} \Lambda_S$.

Let H' be a graph that has a nice embedding mentioned in Lemma 6.7 such that $|V(H')| \leq h^*$. By Lemma 6.3, there do not exist $|V(H')|$ d-free vertices such that every pair of them has distance at least ϕ' under the metric $m_{\mathcal{T}'}$, otherwise, G contains an H-subdivision. So by Theorem 5.6 and Lemma 5.7, there exist integer k with $0 \leq k \leq h^*$, d-free vertices $v_1, v_2, ..., v_k$ of G' with respect to $\mathcal{T}'$, and $(\phi'+10)$-zones $\Lambda_1, \Lambda_2, ..., \Lambda_k$ around $v_1, v_2, ..., v_k$, respectively, such that every d-free vertex of G' with respect to $\mathcal{T}'$ is in the interior of $\bigcup_{i=1}^k \Lambda_i$. Then every d-free vertex in $G-Z$ with respect to $\mathcal{T}-Z$ is a vertex of G', and it is in the interior of $\bigcup_{i=1}^k \Lambda_i \cup \bigcup_{(S,\Omega)\in\mathcal{S}_2} \Lambda_S$.

Then let $\mathcal{S}^* = \mathcal{S}_1^* \cup \mathcal{S}_2^*$, $\mathcal{T}^*$ and G^* be the $\mathcal{S}^*$, $\mathcal{T}^*$ and G^*, respectively, mentioned in the conclusion of Lemma 6.5 by taking $\phi = \phi^*$, $\rho = \rho_{6.6}$, $\lambda = d+\phi'+11$, $\kappa = \kappa_{6.6}$, $k = h^*+\kappa_{6.6}$, $\theta^* = \theta_{6.2}(d, h^*, \rho_{6.6}, \kappa_{6.6}, (dh^*+h^*+1)(\theta_{5.8}+1))$, $d = d$, $s = 1$ and $\mathcal{F}$ be the family of graphs that contains exactly one vertex with no edges, and further taking $G = G-Z$, $\mathcal{T} = \mathcal{T}-Z$, $\mathcal{S} = \mathcal{S}$, $\tau = \tau$, $\Sigma = \Sigma$, and G' to be the skeleton of $\mathcal{S}$.

Let κ' be the number of members of $\mathcal{S}_2^*$ containing d-free vertices with respect to $\mathcal{T} - Z$. Let $Z_1, Z_2, ..., Z_{\kappa'}, U_1, U_2, ..., U_{\kappa'}$ be the sets obtained by applying Lemma 6.2 by taking $h_i = h^*$ for every i, $\rho = \rho_{6.5}$, $\theta'' = (dh^* + h^* + 1)(\theta_{5.8} + 1)$, $G = G - Z$, $G' = G^*$ and $(S_1, \Omega_1), (S_2, \Omega_2), ...$ as the vortices in $\mathcal{S}_2^*$ containing d-free vertices with respect to $\mathcal{T} - Z$. Define $\mathcal{S}_2^{*\prime} \subseteq \mathcal{S}_2^*$ to be consisting of the members in which $U_i \neq \emptyset$. We replace Z by $Z \cup \bigcup_{1 \leq i \leq \kappa'} Z_i$. Note that $|Z| \leq \xi$. If $|\mathcal{S}_2^{*\prime}| = 0$, then there do not exist d-free vertices of $G - Z$ with respect to $\mathcal{T} - Z$, so Statement 1 holds. If $\mathrm{mf}(H, \Sigma) \geq 2$ and $|\mathcal{S}_2^{*\prime}| \leq \mathrm{mf}(H, \Sigma) - 1$, then Statement 2(b) holds. So we may assume that $|\mathcal{S}_2^{*\prime}| \geq \mathrm{mf}(H, \Sigma)$.

Let G'' be the graph and $\mathcal{T}''$ the tangle in G'' of order at least $(dh^* + h^* + 1)(\theta_{5.8} + 6)$ conformal with $\mathcal{T}^*$ mentioned in the conclusion of Lemma 6.2. For $1 \leq i \leq |\mathcal{S}_2^{*\prime}|$ and $1 \leq j \leq h^*$, let Y_i and $A_{i,j}$ be the sets mentioned in Conclusion 2(b) of Lemma 6.2. Since $\mathcal{T}''$ is obtained from $\mathcal{T} - Z$ by deleting at most $\kappa_{6.2}\beta_{6.2}$ vertices and clearing at most $\kappa_{6.2}$ $f_{6.2}(d, h^*, \rho_{6.5}, \kappa_{6.2})$-zones, for every $1 \leq i < i' \leq |\mathcal{S}_2^{*\prime}|$, $j, j' \in \{1, 2, ..., h^*\}$, $x \in A_{i,j}, y \in A_{i',j'}$, we have that $m_{\mathcal{T}''}(x, y) \geq m_{\mathcal{T}-Z}(x, y) - \kappa_{6.2}(4\beta_{6.2} + 2 + \beta_{6.2}) \geq \phi^*(\rho_{6.6}) - 2f_{6.2}(d, h^*, \rho_{6.5}, \kappa_{6.2}) - 4 - \kappa_{6.2}(5\beta_{6.2} + 2) \geq \theta_{5.8} + (dh^* + h^* + 1)(\theta_{5.8} + 6)$.

Let $x \in A_{1,1}$. By Lemma 5.4, there exists an edge e^* of G'' with $m_{\mathcal{T}''}(e^*, x) \geq (dh^* + h^* + 1)(\theta_{5.8} + 6)$. As in the proof of Theorem 4.3 in [12], there exist edges $e_1, e_2, ..., e_{dh^*+h^*}$ in that path such that $(\theta_{5.8} + 6)i \leq m_{\mathcal{T}''}(x, e_i) \leq (\theta_{5.8} + 6)i + 3$ for $1 \leq i \leq dh^* + h^*$. Therefore, $m_{\mathcal{T}''}(e_i, e_j) \geq \theta_{5.8} + 3$ for every $1 \leq i < j \leq dh^* + h^*$, and the set of the ends of each e_i is free for $1 \leq i \leq dh^* + h^*$. Note that $m_{\mathcal{T}''}(x, y) \leq 2$ for $y \in \bigcup_{j=1}^{h^*} A_{1,j}$ and $m_{\mathcal{T}''}(x, y) \geq \theta_{5.8} + (dh^* + h^* + 1)(\theta_{5.8} + 6)$ for $y \in \bigcup_{i=2}^{|\mathcal{S}_2^{*\prime}|} \bigcup_{j=1}^{h^*} A_{i,j}$. Hence, $m_{\mathcal{T}''}(y, e_\ell) \geq \theta_{5.8} + 1$ for every $y \in \bigcup_{i=1}^{|\mathcal{S}_2^{*\prime}|} \bigcup_{j=1}^{h^*} A_{i,j}$ and $1 \leq \ell \leq dh^* + h^*$. For $1 \leq i \leq |\mathcal{S}_2^{*\prime}|$, define Δ_i to be a disk in Σ contained in the disk bounded by Y_i such that $\Delta_i \cap G'' = \bigcup_{j=1}^{h^*} A_{i,j}$. For $1 \leq i \leq dh^* + h^*$, define $\Delta_{|\mathcal{S}_2^{*\prime}|+i}$ to be a disk in Σ such that $\Delta_i \cap G''$ is the set of the ends of e_i. Since H' has a nice embedding in Σ, we can embed H' into Σ such that the vertices of degree at least 4 of H' are incident with $\mathrm{mf}(H', \Sigma)$ regions. Let G''' be the graph obtained from G'' by adding disjoint d-spiders in G from some vertices in $\bigcup_{(S,\Omega) \in \mathcal{S}_2^{*\prime}} S$ to $A_{i,j}$ mentioned in Lemma 6.2 for each $1 \leq i \leq |\mathcal{S}_2^{*\prime}|$ and $1 \leq j \leq h^*$. Consequently, G''' admits an H'-subdivision (π_V, π_E) by concatenating pairwise disjoint d-spiders from some vertices in some members of $\mathcal{S}_2^{*\prime}$ to $\bigcup_{i=1}^{\mathrm{mf}(H',\Sigma)} \bigcup_{j=1}^{h^*} A_{i,j}$ and a disjoint union of 3-spiders and a linear forest obtained by applying Theorem 5.8 by appropriately partitioning $\bigcup_{i=1}^{\mathrm{mf}(H',\Sigma)} \bigcup_{j=1}^{h^*} A_{i,j} \cup \bigcup_{i=1}^{|E(H')|+|V(H')|} \{a_i, b_i\}$, where a_i, b_i are the ends of

e_i.

Finally, we shall prove that G admits an H-subdivision and lead to a contradiction. Recall that $\mathcal{S}^*$ is maximal, so for every $(S, \Omega) \in \mathcal{S}_1^*$ and for every $a \in \bar{\Omega}$, there exist $|\bar{\Omega}|-1$ paths in S from a to $\bar{\Omega}-\{a\}$ intersecting in a and otherwise disjoint. Since H' is triangle-free, one edge the triangle in G''' on $\bar{\Omega}$ is not contained in the image of π_E for each $(S, \Omega) \in \mathcal{S}_1^*$ with $|\bar{\Omega}| = 3$. Therefore, G admits an H-subdivision. $\blacksquare$

References

[1] Z. Dvořák, *A stronger structure theorem for excluded topological minors*, arXiv: 1209.0129v1.

[2] R. Diestel and R. Thomas, *Excluding a countable clique*, J. Combin. Theory Ser. B 76 (1999), 41–67.

[3] M. Grohe and D. Marx, *Structure theorem and isomorphism test for graphs with excluded topological subgraphs*, arXiv:1111.1109v1.

[4] K.-i. Kawarabayasi, S. Norine, R. Thomas, and P. Wollan, *K_6 minors in large 6-connected graphs*, arXiv:1203.2192v1.

[5] C.-H. Liu, *Graph Structures and Well-Quasi-Ordering*, PhD Dissertation, Georgia Institute of Technology, 2014.

[6] C.-H. Liu and R. Thomas, *Well-quasi-ordering graphs by the topological minor relation*, manuscript.

[7] D. Marx, *Important separators and parameterized algorithms*, http://www.cs.bme.hu/∼dmarx/papers/marx-mds-separators-slides.pdf, (2011).

[8] D. Marx and P. Wollan, *Immersions in highly edge connected graphs*, arXiv: 1305.1331v3.

[9] N. Robertson and P. D. Seymour, *Graph minors. IX. Disjoint crossed paths*, J. combin. Theory, Ser. B 49 (1990), 40–77.

[10] N. Robertson and P. D. Seymour, *Graph minors. X. Obstructions to tree-decomposition*, J. combin. Theory Ser. B 52 (1991), 153–190.

[11] N. Robertson and P. D. Seymour, *Graph minors. XI. Circuits on a surface*, J. combin. Theory Ser. B 60 (1994), 72–106.

[12] N. Robertson and P. D. Seymour, *Graph minors. XII. Distance on a surface*, J. combin. Theory Ser. B 64 (1995), 240–272.

[13] N. Robertson and P. D. Seymour, *Graph minors. XIII. The disjoint paths problem*, J. combin. Theory Ser. B 63 (1995), 65–110.

[14] N. Robertson, P. D. Seymour, *Graph minors. XIV. Extending an embedding*, J. Combin. Theory, Ser. B **65** (1995) 23–50.

[15] N. Robertson and P. D. Seymour, *Graph minors. XVI. Excluding a non-planar graph*, J. combin. Theory Ser. B 89 (2003), 43–76.