MINIMALLY NON-PFAFFIAN GRAPHS

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Abstract. We consider the question of characterizing Pfaffian graphs. We exhibit an infinite family of non-Pfaffian graphs minimal with respect to the matching minor relation. This is in sharp contrast with the bipartite case, as Little [7] proved that every bipartite non-Pfaffian graph contains a matching minor isomorphic to $K_{3,3}$. We relax the notion of a matching minor and conjecture that there are only finitely many (perhaps as few as two) non-Pfaffian graphs minimal with respect to this notion.

We define Pfaffian factor-critical graphs and study them in the second part of the paper. They seem to be of interest as the number of near perfect matchings in a Pfaffian factor-critical graph can be computed in polynomial time. We give a polynomial time recognition algorithm for this class of graphs and characterize non-Pfaffian factor-critical graphs in terms of forbidden central subgraphs.

1. Introduction

All graphs in this paper are finite and simple, and cycles and paths have no repeated vertices. A subgraph H of a graph G is *central* if $G \setminus V(H)$ (we use $\setminus$ for deletion and $-$ for set theoretic difference) has a perfect matching. An even cycle C in a directed graph D is called *oddly* (resp. *evenly*) *oriented* if for either choice of direction of traversal around C, the number of edges of C directed in the direction of traversal is odd (resp. even). An orientation D of a graph G with an even number of vertices is called *Pfaffian* if every central cycle C of G is oddly oriented in D. A graph G with an even number of vertices is said to be *Pfaffian* if it admits a Pfaffian orientation. The significance of this notion stems from the fact that if a graph G is Pfaffian, then the number of perfect matchings of G, and, more generally, the generating function of perfect matchings, can be computed in polynomial time. This was discovered by Kasteleyn [4, 5, 6] and Fisher [3] and has received considerable attention since then. We refer to [16] for a recent survey.

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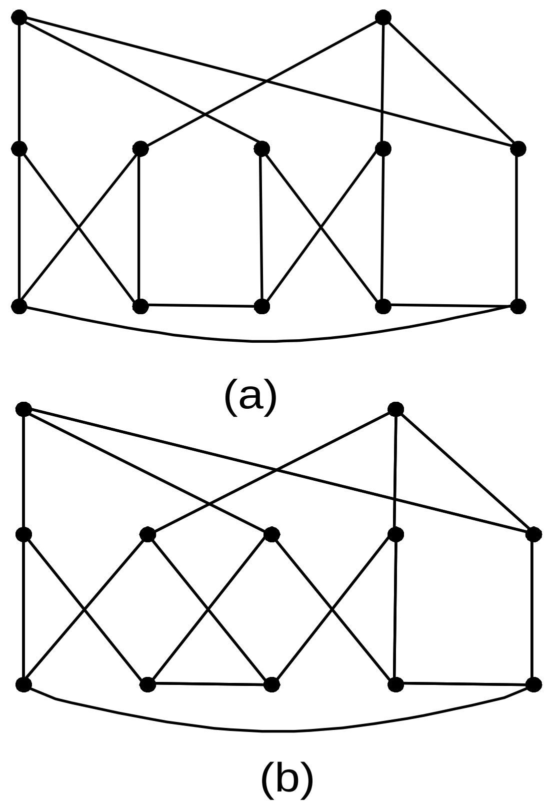

FIGURE 1. (a) Cubeplex, (b) Twinplex.

In this paper we address the question of characterizing Pfaffian graphs. The following theorem of Little [7] gives an elegant characterization of bipartite Pfaffian graphs. Let H be a graph, and let v be a vertex of H of degree two. By *bicontracting* v we mean contracting both edges incident with v and deleting the resulting loops and parallel edges. A graph G is a *matching minor* of a graph H if G can be obtained from a central subgraph of H by repeatedly bicontracting vertices of degree two. It is fairly easy to see that a matching minor of a Pfaffian graph is Pfaffian.

Theorem 1.1. *A bipartite graph admits a Pfaffian orientation if and only if it has no matching minor isomorphic to* $K_{3,3}$.

Does there exist an analogue of Theorem 1.1 for general graphs? In [2] Fischer and Little extend Theorem 1.1 to a larger class of graphs. Let us give the necessary definitions before stating their result. A graph in which every edge belongs to a perfect matching is said to be *matching-covered*. A matching-covered non-bipartite graph G is *near-bipartite* if there exist $e, f \in E(G)$ such that $G \setminus \{e, f\}$ is matching-covered and bipartite. A graph H is said to be a *weak matching minor* of a graph G if H can be obtained from a matching minor of G by a sequence of odd cycle contractions. (When contracting odd cycles the resulting loops and parallel edges are deleted.) It is shown in [8] that the property of being Pfaffian is closed under taking weak matching minors. Cubeplex and twinplex are particular graphs on 12 vertices (see Figure 1). The following theorem of Fischer and Little [2] gives a characterization of near-bipartite Pfaffian graphs in terms of forbidden weak matching minors.

Theorem 1.2. *A near-bipartite graph is Pfaffian if and only if it has no matching minor isomorphic to $K_{3,3}$, cubeplex or twinplex.*

Let us say that a graph G is *minimally non-Pfaffian* if it is not Pfaffian, but every proper weak matching minor of G is Pfaffian. Thus $K_{3,3}$, twinplex and cubeplex are minimally non-Pfaffian, and by Theorem 1.2 they are the only minimally non-Pfaffian near-bipartite graphs. The Petersen graph is also minimally non-Pfaffian. Little (private communication) made the plausibly-looking conjecture that the graphs listed in Theorem 1.2 and the Petersen graph are the only minimally non-Pfaffian graphs; in other words, that Theorem 1.2 holds for all graphs as long as the Petersen graph is added to the list of excluded weak matching minors. Unfortunately, that is not true. In Section 3 we exhibit an infinite family of minimally non-Pfaffian graphs.

The structure of the family suggests several reduction operations that preserve the Pfaffian property. We describe these operations in Sections 2 and 4, and use them in Section 4 to formulate a modified conjecture that includes only two obstacles rather than infinitely many.

We then turn to factor-critical graphs. A graph G is *factor-critical* if $G \setminus v$ has a perfect matching for every vertex $v \in V(G)$. For $u \notin V(G)$ we define G^u to be the graph obtained from G by adding a vertex u joined by an edge to every vertex of G. We say that a graph G with $|V(G)|$ odd is *Pfaffian* if G^u is Pfaffian. We say that a matching of G is *near-perfect* if it covers all but one vertex of G. Similarly as for graphs with Pfaffian orientations, if a factor-critical graph is Pfaffian, then the number of near-perfect matchings in G can be enumerated in polynomial time. In Section 5 we design a polynomial-time algorithm to test if a factor-critical graph is Pfaffian, and in Section 6 we prove an analogue of Theorem 1.1 for factor-critical graphs.

2. New operations that preserve the Pfaffian property of the graph

In Section 3 we will exhibit a family of non-Pfaffian graphs such that no element of this family can be reduced to a smaller non-Pfaffian graph by edge deletion, bicontraction or contraction of an odd cycle. This motivates a search for other reduction operations that preserve the Pfaffian property. In this section we define such an operation, namely "compression". Further variants of this operation will be defined in Section 4. We also define "flip" and "closure" operations. These operations can not be considered as reduction operations, but they will be used in the proofs in Sections 3 and 4. We start the section with a definition and two preliminary lemmas.

The following alternative definition of Pfaffian orientations will be useful in our analysis of the above operations. Let D be an orientation of a graph G. We say that two perfect matchings have *the same sign* in D if their symmetric difference contains an even number of evenly oriented cycles. Clearly, an orientation of the graph is Pfaffian if and only if every two perfect matchings have the same sign in it. Having the same sign is an equivalence relation [9], and we will refer to an equivalence class of a perfect matching as its *sign*.

Lemma 2.1. *Let G be a graph, let D be a Pfaffian orientation of G, let $u, v \in V(G)$ be not adjacent, let C be a cycle in $G+uv$ and let $uv \in E(C)$. If every perfect matching M of $G+uv$, such that $uv \in M$, can be transformed to a perfect matching M' of $G+uv$ such that C is M'-alternating by repeatedly taking symmetric difference of M with M-alternating circuits of G that are oddly oriented in D then $G+uv$ is Pfaffian.*

Proof. Notice that every perfect matching of G is a perfect matching of $G+uv$. Let D' be an orientation of $G+uv$ obtained from D by orienting uv in such a way that C is oddly oriented. We claim that D' is Pfaffian. It suffices to show that every perfect matching M of $G+uv$ has the same sign as some perfect matching of G. Let a perfect matching M' of G such that C is M'-alternating be constructed from M as in the statement of the lemma. Then M' and M have the same sign in D' as taking symmetric difference of a perfect matching with an oddly oriented circuit does not change its sign. Finally M' has the same sign as a perfect matching $M \triangle C$ of G. □

Lemma 2.2. *Let G be a connected Pfaffian graph, and let T be a spanning tree of G. Then an arbitrary orientation of T extends to a Pfaffian orientation of G. Furthermore, if $e \in E(G)$ joins two vertices at even distance in T then an arbitrary orientation of $T+e$ extends to a Pfaffian orientation of G.*

Proof. An orientation obtained from a Pfaffian orientation by reversing direction of all edges in a cut is Pfaffian. This observation immediately implies the first statement of the lemma. An orientation obtained from a Pfaffian orientation by reversing direction of every edge is also Pfaffian. To show that the second statement holds, let us color the vertices of the graph in two

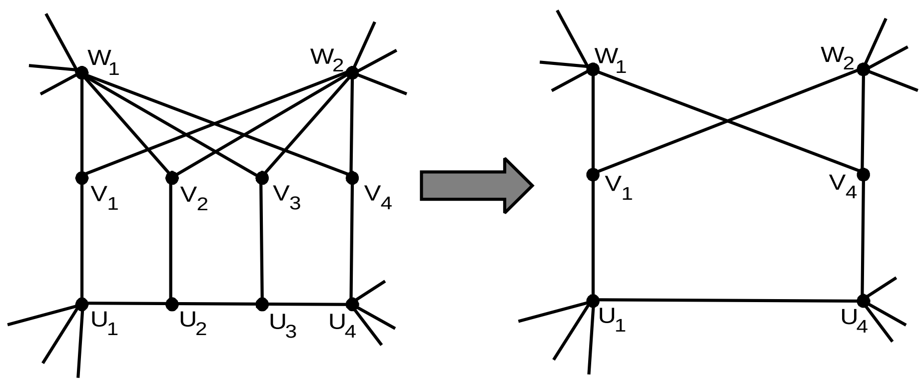

FIGURE 2. Compression of H.

colors, so that the coloring of T is proper. Given such a coloring, we reverse the direction of every edge in the graph and then we reverse the direction of all the edges in the cut separating the color classes. In the resulting graph, the orientation of every edge in T remains unchanged, while the direction of e is reversed. Therefore the second statement of the lemma follows from the first. □

Consider a graph G containing a central subgraph H such that

$$V(H) = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, w_1, w_2\},$$

$$E(H) = \{u_1u_2, u_2u_3, u_3u_4\} \cup \bigcup_{i=1}^{4} \{u_iv_i, v_iw_1, v_iw_2\},$$

and the degree in G of each of the vertices $u_2, u_3, v_1, v_2, v_3, v_4$ is three. We form a graph G' from G as follows: delete the vertices u_2, u_3, v_2, v_3 from G, and add an edge u_1u_4 (see Figure 2). We say that G' is obtained from G by *a compression* (of H). In Section 4 we will define two similar operations, which will be referred to as compressions of types two and three.

Lemma 2.3. *Let G be a Pfaffian graph and let G' be obtained from G by compression of a subgraph H of G. Then G' is Pfaffian.*

Proof. Let the vertices of H be labeled as in the definition above. We claim that the graph $G+u_1u_4$ is Pfaffian. Let $C = u_1v_1w_1v_3u_3u_4u_1$. For a perfect matching M of $G + u_1u_4$ such that $u_1u_4 \in M$ we have either $v_1w_1 \in M$, in which case C is M-alternating, or $v_1w_2 \in M$, in which case $v_4w_1 \in M$ and M can be transformed to a perfect matching $M \triangle C_{1,4}$ of $G + u_1u_4$ containing u_1u_4 and v_1w_1, where $C_{i,j}$ is the edge-set of the cycle $v_iw_1v_jw_2v_i$ for $1 \leq i < j \leq 4$. To derive our claim from Lemma 2.1 it suffices to show that $C_{1,4}$ is oddly oriented in some Pfaffian orientation of G. Note that

$$C_{1,4} \triangle C_{1,2} \triangle C_{2,3} \triangle C_{3,4} = \emptyset.$$

Therefore the parity of the number of oddly oriented circuits among $C_{1,4}$, $C_{1,2}$, $C_{2,3}$ and $C_{3,4}$ is independent of the choice of orientation and is even. Finally note that $C_{1,2}, C_{2,3}$ and $C_{3,4}$ are central in H and therefore in G and as such are oddly oriented in any Pfaffian orientation of G. □

We would like to prove a converse of Lemma 2.3. This result, in particular, will be used in the following section.

Lemma 2.4. *Let a Pfaffian graph G' be obtained from a graph G by compression of a subgraph H of G. Then G is Pfaffian.*

Proof. Label the vertices of H as in the definition of compression. Consider a Pfaffian orientation D of G'. By Lemma 2.2 we may assume that w_1v_1, w_1v_4, v_4w_2, v_1u_1, $v_4u_4, u_1u_4 \in E(D)$. Since the cycle $v_1w_2v_4w_1$ is central in G' it follows that $w_2v_1 \in E(D)$. We extend D to an orientation D' of G as follows: w_1v_2, w_1v_3, w_2v_3, v_2w_2, v_2u_2, v_3u_3, u_1u_2, u_3u_2, $u_3u_4 \in E(D')$. We claim that D' is a Pfaffian orientation of $G_+ = G + u_1u_4$. It suffices to prove that every perfect matching M of G_+ has the same sign in D' as some perfect matching of G_+ containing u_2v_2 and u_3v_3.

Suppose first that $u_2v_2 \in M$. Then we can assume that $w_iv_3 \in M$ for some $i \in \{1, 2\}$. We have $v_1u_1, u_3u_4 \in M$ and by taking the symmetric difference of M with the oddly oriented cycle $u_1v_1w_iv_3u_3u_4u_1$ we get a perfect matching M' that contains u_2v_2 and u_3v_3 and has the same sign as M, as desired. The case when $u_3v_3 \in M$ is symmetric. In the only remaining case $u_2u_3, u_1v_1, u_4v_4 \in M$ and $v_2w_i, v_3w_j \in M$ for some $\{i, j\} = \{1, 2\}$. We consider the symmetric difference of M with the oddly oriented cycle $u_1v_1w_iv_2u_2u_3v_3w_jv_4u_4u_1$ to verify the claim for M. □

We now define the second of our operations. Suppose a graph G contains a central subgraph H such that

$$V(H) = \{u_1, u_2, v_1, v_2, w_1, w_2\},$$

$$E(H) = \{u_1u_2, u_1v_1, u_2v_2, v_1w_1, v_1w_2, v_2w_1, v_2w_2\},$$

and the degree in G of the vertices v_1 and v_2 is three. Then we say that H is a *fin*. We form a graph G' from G as follows: delete the edges v_1w_2, v_2w_1, u_1u_2 from G, and add the edges v_1u_2, v_2u_1 and w_1w_2 (see Figure 3). We say that G' is obtained from G by a *flip* (of the fin H). Note that in this case G can be obtained from G' by a flip of a fin on the same vertex set as H.

Lemma 2.5. *Let G be a Pfaffian graph and let G' be obtained from G by a flip of a fin H. Then G' is Pfaffian.*

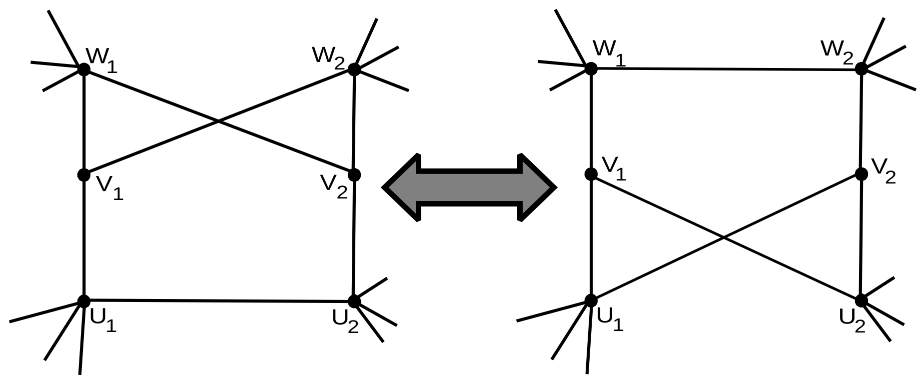

FIGURE 3. A flip.

Proof. Let the vertices of H be labeled as in the definition of a flip. By Lemma 2.1 the graph $G_* = G + w_1w_2$ is Pfaffian (consider the cycle $C = w_1v_1u_1u_2v_2w_2w_1$). Let D be a Pfaffian orientation of G_*. By Lemma 2.2 we can assume that w_1v_1, w_1v_2, v_1w_2, w_2v_2, w_1w_2, u_1v_1, u_2v_2, $u_1u_2 \in E(D)$. It follows that $w_1w_2 \in E(D)$, because the cycle $w_1w_2v_2u_2u_1v_1w_1$ is central, and that $v_1w_2 \in E(D)$, because the cycle $w_1v_2w_2v_1w_1$ is central. Let an orientation D' of $G_* + v_2u_1 + u_2v_1$ be obtained from D by setting $v_2u_1, u_2v_1 \in D'$. We claim that D' restricted to G' is a Pfaffian orientation. It suffices to show that any perfect matching M of G' has the same sign as some perfect matching of G_* in D'.

Suppose that $u_1v_2 \in M$. Then either $v_1u_2 \in M$ or $v_1w_1 \in M$. In the first case M and $M \triangle C$ have the same sign where $C = u_1v_1u_2v_2$, while in the second case M and $M \triangle C'$ have the same sign where $C' = u_1v_1w_1v_2$. The case $u_2v_1 \in M$ is analogous. □

Let G be a graph and let $u, v \in V(G)$. We say that $G + uv$ is a *closure* of G if every central cycle C in $G \setminus \{u, v\}$ is central in G.

Lemma 2.6. *A closure of a Pfaffian graph is Pfaffian.*

Proof. The symmetric difference of any two perfect matchings M_1, M_2 of $G + uv$ such that $uv \in M_1 \cap M_2$ is a union of cycles that are oddly oriented in any Pfaffian orientation of G. Therefore a Pfaffian orientation of $G + uv$ can be obtained from a Pfaffian orientation of G by orienting uv in such a way that some perfect matching of $G + uv$ containing uv has positive sign. □

3. A FAMILY OF MINIMALLY NON-PFAFFIAN GRAPHS

Let $k \geq 1$ be an integer, let C_{2k+1} be the cycle of length $2k + 1$ with vertices labeled $1, 2, \ldots, 2k + 1$, in order, and let M be a matching in C, possibly empty. The graph $G(k, M)$ is defined as follows. Let

$$V(G(k, M)) = \{u_1, u_2, \ldots, u_{2k+1}, v_1, v_2, \ldots, v_{2k+1}, w_1, w_2\}$$

and let $G(k, M)$ have the following edges, where the indices are considered modulo $2k + 1$:

- $u_i v_i$ for every $1 \le i \le 2k + 1$,
- $u_i u_{i+1}, v_{i+1} w_1$ and $v_i w_2$ if $\{i, i + 1\} \notin M$,
- $u_i v_{i+1}$ and $v_i u_{i+1}$ if $\{i, i + 1\} \in M$.

The vertices $u_1, u_2, \ldots, u_{2k+1}, v_1, v_2, \ldots v_{2k+1}$ have degree three in $G(k, M)$, while w_1 and w_2 have degree $2k + 1 - |M|$. The cubeplex graph shown on Figure 1 is isomorphic to the graph $G(2, \{\{1, 2\}, \{3, 4\}\})$, and is, up to an isomorphism, the unique cubic graph in the family described above, as well as the unique such near-bipartite graph. We will prove that the graph $G(k, M)$ is minimally non-Pfaffian for $k \ge 2$.

We will use the following lemma in our analysis. It follows from [9, Theorem 8.3.7].

Lemma 3.1. *A graph G is non-Pfaffian if and only if there exist an orientation D of G and central cycles $C_1, C_2, \ldots, C_k$ of G for some $k \ge 1$ such that $\triangle_{i=1}^{k} C_i = \emptyset$ and odd number of cycles in the family $C_1, C_2, \ldots, C_k$ are evenly oriented in D.*

Theorem 3.2. *The graph $G(1, \emptyset)$ is non-Pfaffian. The graph $G(k, M)$ is non-Pfaffian for every integer $k \ge 2$ and every matching M of C_{2k+1}.*

Proof. The proof is by induction on $|M|$. We start by considering $M = \emptyset$. The graph $G(k, \emptyset)$ can be reduced by a sequence of compressions to the graph $G(1, \emptyset)$ obtained from $K_{3,3}$ by replacing one of its vertices by a triangle. Therefore, $G(k, \emptyset)$ for $k \ge 1$ is non-Pfaffian by Lemma 2.4.

Suppose now $M \ne \emptyset$. Denote $G(k, M)$ by G for brevity. Choose i so that $\{i, i+1\} \in M$. The graph $G + w_1 w_2$ can be obtained from $G(k, M - \{i, i+1\})$ by flipping the fin induced on $\{u_i, u_{i+1}, v_i, v_{i+1}, w_1, w_2\}$. Therefore by the induction hypothesis and Lemma 2.5 the graph $G + w_1 w_2$ is non-Pfaffian. Moreover, $G + w_1 w_2$ is a closure of G. It now follows from Lemma 2.6 that G is non-Pfaffian. □

Theorem 3.3. *The graph $G(k, M)$ is minimally non-Pfaffian for every $k \ge 2$ and every matching M of C_{2k+1}.*

Proof. By Theorem 3.2 it suffices to prove that every graph obtained from $G(k, M)$ by deleting an edge or contracting an odd cycle is Pfaffian. (We do not need to consider bicontractions, as the graph $G(k, M)$ has minimum degree three.)

We start by proving by induction on $|M|$ that $G(k, M) \setminus e$ is Pfaffian for every $e \in E(G(k, M))$. We consider the base case $M = \emptyset$ first. For $1 \leq i \leq 2k+1$ let M_{2i-1} be the perfect matching of $G(k, \emptyset)$ consisting of edges $u_i u_{i+1}$, $v_i w_1$, $v_{i+1} w_2$ and $u_j v_j$ for all $1 \leq j \leq 2k+1$ such that $j \notin \{i, i+1\}$. For $1 \leq i \leq 2k+1$ let the perfect matching M_{2i} be obtained from M_{2i-1} by replacing edges $v_i w_1$ edges $v_{i+1} w_2$ with $v_i w_2$, $v_{i+1} w_1$. The set $\{M_1, M_2, \ldots, M_{4k+2}\}$ is the set of all perfect matchings of $G(k, \emptyset)$. For every $1 \leq i < 4k+2$ there exists $e \in E(G(k, \emptyset))$ such that $e \in M_i \cap M_{i+1}$, and e does not lie in any other perfect matching of G. It follows that $\triangle_{i \in S} M_S \neq \emptyset$ for every proper non-empty subset S of $\{1, \ldots, 4k+2\}$. Thus, by Lemma 3.1, we have that $G(k, \emptyset) \setminus e$ is Pfaffian for every $e \in E(G(k, \emptyset))$.

For the induction step denote $G(k, M)$ by G for brevity and suppose first that there exists $\{i, i+1\} \in M$ such that

$$e \notin \{u_i v_i, u_i v_{i+1}, u_{i+1} v_i, u_{i+1} v_{i+1}, v_i w_1, v_{i+1} w_2\}.$$

If the graph $G \setminus e \setminus \{u_i, v_i, u_{i+1}, v_{i+1}, w_1, w_2\}$ has a perfect matching then $(G \setminus e) + w_1 w_2$ can be obtained by a flip from $G(k, M - \{i, i+1\}) \setminus e$. Therefore $G \setminus e$ is Pfaffian by the induction hypothesis and Lemma 2.5. If, on the other hand, $G \setminus e \setminus \{u_i, v_i, u_{i+1}, v_{i+1}, w_1, w_2\}$ has no perfect matching then $e = u_j v_j$ for some j unsaturated by M. The graph $G \setminus u_j v_j$ has an independent set $W = \{v_1, \ldots, v_{2k+1}, u_j\}$ with $|W| = 2k+2 = |V(G)|/2$. Choose $f = u_l u_{l+1} \in E(G)$ such that $j \notin \{l, l+1\}$. Then f has no end in W and therefore f lies in no perfect matching of $G \setminus u_j v_j$. It follows that $G \setminus u_j v_j$ is Pfaffian if $G \setminus \{u_j v_j, f\}$ is Pfaffian, and we have already shown that $G \setminus f$ is Pfaffian.

It remains to consider the case when the choice of i made above is impossible. In this case $|M| = 1$. Without loss of generality we assume $M = \{1, 2\}$ and $e \in \{u_1 v_2, v_1 w_1\}$. If $e = u_1 v_2$ then the edge $v_1 u_2$ lies in a unique perfect matching of $G \setminus e$ and it suffices to show that $G \setminus \{e, v_1 u_2\}$ is Pfaffian. But it is Pfaffian because it is a proper subgraph of $G(k, \emptyset)$. Finally, if $e = v_1 w_1$ then the edge $u_2 u_3$ lies in a unique perfect matching of $G \setminus e$. It follows that $G \setminus e$ is Pfaffian because we have shown above that $G \setminus u_2 u_3$ is Pfaffian.

We have proven that $G(k, M) \setminus e$ is Pfaffian for every $k \geq 2$ and every $e \in E(G(k, M))$. It remains to show that every graph G' obtained from $G(k, M)$ by contracting an odd cycle C is Pfaffian. By the above we may assume that C is induced and no vertex in $G(k, M) \setminus V(C)$ has more than one neighbor in $V(C)$. Otherwise, G' can be obtained from a proper subgraph of $G(k, M)$ by contracting C, and hence is Pfaffian.

Suppose first that $w_1, w_2 \notin V(C)$. Then $u_1, \ldots, u_{2k+1} \in V(C)$ as the graph $G(k, M) \setminus \{w_1, w_2, u_j\}$ is bipartite for every $1 \leq j \leq 2k+1$. For every $\{i, i+1\} \in M$ exactly one of the vertices v_i and v_{i+1} lies in C, while the other has two neighbors in $V(C)$. It follows that $M = \emptyset$, and hence $V(C) = \{u_1, u_2, \ldots, u_{2k+1}\}$. It follows that G' is isomorphic to $K_{3,2k+1}$, and hence is Pfaffian, because it has no perfect matching.

Therefore we may assume that $\{w_1, w_2\} \cap V(C) \neq \emptyset$ and without loss of generality we assume $w_1v_1, v_1u_1 \in E(C)$. If $u_1v_2 \in E(C)$ then u_2 has two neighbors $v_1, v_2 \in V(C)$ and thus $v_2u_2 \in E(C)$, contradicting the assumption that C is induced. Therefore without loss of generality we may assume that $u_1u_2 \in E(C)$. The vertex v_2 has two neighbors $w_1, u_2 \in V(C)$ and consequently $C = w_1v_1u_1u_2v_2w_1$. The vertex w_2 has at most one neighbor in C and therefore $\{2, 3\} \in M$. Note that a graph isomorphic to G' may be obtained by contraction of C in the Pfaffian graph $G(k, M - \{2, 3\}) \setminus w_2v_2$. Therefore G' is Pfaffian. □

Conjecture 3.4. *Every minimally non-Pfaffian graph is isomorphic to $K_{3,3}$, twinplex, the Petersen graph, or the graph $G(k, M)$ for some integer $k \geq 2$ and some matching M of C_{2k+1}.*

4. Revised conjecture

We would like to restate Conjecture 3.4 in a way that involves two obstructions, rather than infinitely many. To do this we need to expand our set of reduction operations. As mentioned in the proof of Theorem 3.2 the graph $G(k, \emptyset)$ can be reduced to $K_{3,3}$ by a sequence of compressions and an odd cycle contraction, and the graph $G(k, M)$ can be reduced to $G(k, \emptyset)$ by a sequence of flips and closures. The flip and closure operations, however, do not seem to be natural reduction operations, and so in this section we introduce two additional operations that produce smaller Pfaffian graphs from larger Pfaffian graphs. The operations will be referred to as compressions of type two and three, and we will refer to the compression operation defined in Section 2 as compression of type one. Compressions of types one, two and three can be used to reduce any graph in the family $G(k, M)$ to the graph $G(1, \emptyset)$.

Consider a graph G containing a central subgraph H such that

$$V(H) = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, w_1, w_2\},$$

$$E(H) = (\{u_1u_2, u_3u_4, v_2u_3, u_2v_3\} \cup \bigcup_{i=1}^{4} \{u_iv_i, v_iw_1, v_iw_2\}) - \{v_3w_1, v_2w_2\},$$

the degree in G of each of the vertices $u_2, u_3, v_1, v_2, v_3, v_4$ is three (see Figure 4a), and the cycle $v_2u_2v_3u_3v_2$ is central in G. We form a graph G' from G as follows: delete the vertices u_2, u_3, v_2, v_3 from G, and add the edge u_1u_4. We say that G' is obtained from G by *a compression of type two* (of H).

Lemma 4.1. *Let G be a Pfaffian graph and let G' be obtained from G by a compression of type two of a subgraph H of G. Then G' is Pfaffian.*

Proof. The first part of the proof parallels the proof of Lemma 2.3. Assume that the vertices of H are labeled as in the definition of a compression of type two. We claim that the graph $G + w_1w_2$ is Pfaffian. Let $C = w_1v_1u_1u_2v_3w_2w_1$. For a perfect matching M of $G+w_1w_2$ such that $w_1w_2 \in M$ we have $v_1u_1 \in M$ and either $u_2v_3 \in M$, in which case C is M-alternating, or $u_2v_2 \in M$, in which case $v_2u_3 \in M$ and M can be transformed to a perfect matching $M \triangle C'$ where $C' = v_2u_2v_3u_3$. Our claim follows now from Lemma 2.1 as C' is central in G by definition of compression of type two, and as such is oddly oriented in any Pfaffian orientation of G. The graph G' can be obtained from the graph $G + w_1w_2$ by flipping the fin induced on the vertex set $\{w_1, w_2, v_2, v_3, u_2, u_3\}$, and a compression of type one of the resulting subgraph induced on $V(H)$. Therefore the lemma follows from Lemmas 2.3 and 2.5. □

We now introduce our last reduction operation. Consider a graph G containing a central subgraph H such that

$$V(H) = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, w_1, w_2\},$$

$$E(H) = \{u_1v_2, u_3v_4, v_2u_1, u_2u_3, v_1w_1, v_3w_1, v_2w_2, v_4w_2\} \cup \bigcup_{i=1}^{4}\{u_iv_i\},$$

the degree in G of each of the vertices $u_2, u_3, v_1, v_2, v_3, v_4$ is three (see Figure 4b), and the cycles $v_1u_1v_2u_2v_1$ and $v_3u_3v_4u_4v_3$ are central in G. We form a graph G' from G as follows: delete the vertices u_2, u_3, v_2, v_3 from G, and add the edges u_1u_4, w_1v_4 and v_1w_2. We say that G' is obtained from G by *a compression of type three* (of H).

Lemma 4.2. *Let G be a Pfaffian graph and let G' be obtained from G by compression of type three of a subgraph H of G. Then G' is Pfaffian.*

Proof. We assume that the vertices of H be labeled as in the definition. of a compression of type three. As in Lemma 4.1 our first goal is to prove that the graph $G + w_1w_2$ is Pfaffian. Consider a perfect matching M of $G + w_1w_2$. By taking symmetric differences with central cycles $v_1u_1v_2u_2v_1$ and $v_3u_3v_4u_4v_3$, if necessary, we obtain a perfect matching M' from M such that $v_iu_i \in M$ for all $1 \leq i \leq 4$. The cycle $w_1v_1u_1v_2u_2u_3v_3u_4v_4w_2w_1$ is M'-alternating. Therefore $G + w_1w_2$ is Pfaffian by Lemma 2.1. Obtain a graph G'' from $G + w_1w_2$ by flipping the fin induced on the vertex set $\{w_1, w_2, v_1, v_2, u_1, u_2\}$. By Lemma 2.5 the graph G'' is Pfaffian and by the argument similar to the above (using the cycle $w_1w_2v_4u_4v_3u_2v_2$) the graph $G'' + w_1w_2$ is also Pfaffian. The graph G' can be obtained from the graph

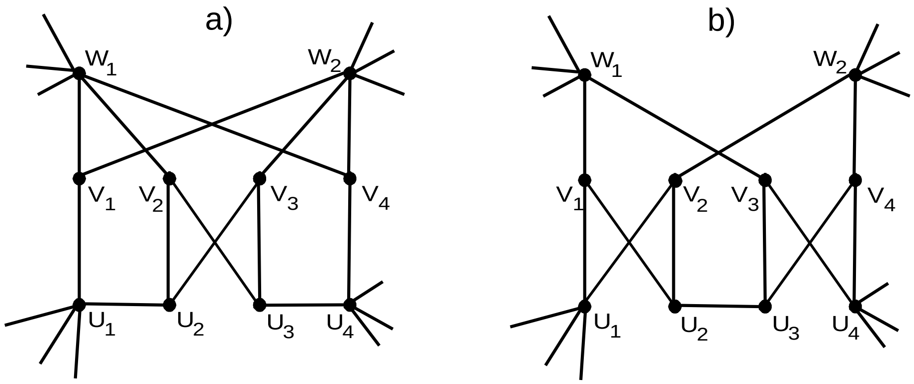

FIGURE 4. Subgraphs compressed by compressions of type two (a), and type three (b).

$G''+w_1w_2$ by flipping the fin induced on the vertex-set $\{w_1, w_2, v_3, v_4, u_3, u_4\}$ and compression of type one of the resulting subgraph induced on $V(H)$, and thus the lemma follows from Lemmas 2.3 and 2.5. $\square$

Consider the graph $G(k, M)$ for some integer $k \geq 2$. If M contains two edges that are at distance one in C_{2k+1}, then compression of type three reduces $G(k, M)$ to the graph $G(k-1, M')$ for some matching M' of C_{2k-1}, where $|M'| = |M| - 2$. If M is non-empty, but does not contain two such edges then compression of type two may be used to reduce $G(k, M)$ to the graph $G(k-1, M')$ for some matching M' of C_{2k-1}, where $|M'| = |M| - 1$. Finally, the graph $G(k, \emptyset)$ for $k \geq 2$ can be reduced to the graph $G(k-1, \emptyset)$ by compression of type one. Thus every graph $G(k, M)$ can be reduced to the non-Pfaffian graph $G(1, \emptyset)$ by a sequence of compressions of types one, two and three. A graph isomorphic to $K_{3,3}$ can be obtained from the graph $G(1, \emptyset)$ by contracting a triangle. The only minimally non-Pfaffian graphs that we know that do not belong to the family $G(k, M)$ are twinplex and the Petersen graph. Moreover, every graph obtained from twinplex by an edge addition or replacement of one of its vertices by a triangle contains a graph isomorphic to $K_{3,3}$ as a matching minor.

Thus we feel tempted to state the following conjecture.

Conjecture 4.3. *A connected graph G is Pfaffian if and only if G is not isomorphic to twinplex and G can not be reduced to $K_{3,3}$ or the Petersen graph by a sequence of edge deletions, bicontractions, contractions of odd cycles and compressions of type one, two and three.*

We have convinced ourselves that Conjecture 4.3 is equivalent to Conjecture 3.4, but the proof of the equivalence is uninteresting, and we omit it. Although we do have some evidence in support of Conjecture 4.3, the compression operations are motivated by the structure of the family from Section 3, and as such seem artificial. A weaker, but perhaps more natural conjecture would state that there exists some finite set of "nice" reduction rules so that every non-Pfaffian graph can be reduced to a graph in some finite collection of non-Pfaffian graphs via repeated application of these reduction rules. Let us state this weaker conjecture precisely.

Let H, H' be graphs with $|E(H')| < |E(H)|$, let $\mathcal{V}$ be a (possibly empty) collection of subsets of $V(H)$, and let $f : W \to V(H')$ be a map for some $W \subseteq V(H)$. Then we say that $\mathcal{R} = (H, H', \mathcal{V}, W, f)$ is *a rule.* Let G be a graph let ϕ be an isomorphism between H and a subgraph of G such that no vertex in $\phi(V(H) - W)$ is incident to an edge in $E(G) - \phi(E(H))$, and for every $V \in \mathcal{V}$ the subgraph of G induced on $\phi(V)$ is central in G. We obtain G' from G by deleting the vertices in $\phi(V(H) - W)$ and edges in $\phi(E(H))$, adding a disjoint copy of H' to the resulting graph, and for every $w \in W$ identifying the vertices $\phi(w)$ and $f(w)$. Then we say that the graph G' is obtained from the graph G by *a reduction using rule* $\mathcal{R}$.

Let us give a simple example. Let H be a path on three vertices, let $\mathcal{V}$ be empty, and let W consist of the ends of the path H. Let H' be a graph with one vertex and let f map W to this vertex. Let $\mathcal{R}_b = (H, H', \mathcal{V}, W, f)$. Then bicontraction can be considered as a reduction using rule $\mathcal{R}_b$.

We say that the rule $\mathcal{R}$ is *valid* if every graph that can be obtained from a Pfaffian graph by a reduction using rule $\mathcal{R}$ is Pfaffian. It might be preferable to define the validity of a rule intrinsically, but at this point it does not seem to be worth the effort. Note that edge deletion, bicontraction, and compressions of type one, two and three can be considered as reductions using valid rules. Thus Conjecture 4.3 implies

Conjecture 4.4. *There exists a finite collection of valid rules such that every non-Pfaffian graph can be reduced to a graph isomorphic to $K_{3,3}$ by repeated reductions using rules from this collection.*

5. Pfaffian factor-critical graphs

In this section we study Pfaffian factor-critical graphs. In particular, we present a polynomial time recognition algorithm for such graphs. We start by introducing the tools that we will use in our proofs.

An *ear-decomposition* of G is a sequence $(C, P_1, \ldots, P_k)$, where C is a central cycle in G and P_i is an odd path that has both ends in and is otherwise disjoint from $C \cup P_1 \cup \ldots \cup P_{i-1}$ for every $i \in \{1, \ldots, k\}$. We use the following structure theorem of Lovász and Plummer [9].

Theorem 5.1. *Let G be a 2-connected factor-critical graph. Then for some integer $k \geq 0$ there exists an ear decomposition $(C, P_1, \ldots, P_k)$ of G. Moreover, $G_i = C \cup P_1 \cup \ldots \cup P_{i-1}$ is a central 2-connected factor-critical subgraph of G for every $i \in \{1, \ldots, k\}$.*

We will also need two lemmas, the first of which is by Pulleyblank [14].

Lemma 5.2. *A graph is factor-critical if and only if it is connected and each of its blocks is factor-critical.*

Lemma 5.3. *Let G be a factor-critical graph, B be a block of G and H be a subgraph of B. Then H is central in G if and only if H is central in B. In particular B is central in G.*

Proof. Let C be a component of $G \setminus V(B)$. Then C is a component of $G \setminus v$ for some $v \in V(B)$. Therefore C has a perfect matching. If H is central in B then a perfect matching of $G \setminus V(H)$ can be obtained by taking union of perfect matching of $B \setminus V(H)$ with perfect matchings of components of $G \setminus V(B)$.

Let now H be a subgraph of B that is central in G and let M be a perfect matching of $G \setminus V(H)$. Since each component of $G \setminus V(B)$ is even we deduce that no edge of M has exactly one end in $V(B)$. Thus $M \cap E(B)$ is a perfect matching of $B \cap V(H)$. □

Our first characterization of Pfaffian factor-critical graphs follows.

Lemma 5.4. *A factor-critical graph G is Pfaffian if and only if there exists an orientation of G in which every central path of length 2 is directed.*

Proof. Assume first that G is Pfaffian. Thus G^u is Pfaffian and so by Lemma 2.2 it has a Pfaffian orientation D such that all the edges incident with u are directed away from u. We claim that the restriction of D to G is as desired. Indeed if $v_0v_1v_2$ is a central path in G, then it is directed as the cycle $uv_0v_1v_2u$ is central in G^u and therefore must be oddly oriented.

Now assume that G has an orientation D' such that every central path of length two is directed, and let D be the orientation of G^u obtained by directing all the edges incident with u away from u. We will prove that D is a Pfaffian orientation of G^u.

Let $C = v_0v_1v_2 \ldots v_{2l+1}v_0$ be a central cycle in G^u and let M be a perfect matching in $G^u \setminus V(C)$. Suppose first that $u = v_0 \in V(C)$. Then for every $i \in \{1, 2, \ldots, l\}$ the path $v_{2i-1}v_{2i}v_{2i+1}$ is central in G and hence is directed in D. It follows that C is oddly oriented. Now suppose that

$u \notin V(C)$ and let $vu \in M$. Let M' be a perfect matching of $G \setminus v_0$. The component of $G[M \cup M']$ containing v is an M-alternating path $P = vu_1u_2 \dots u_{2s-1}u_{2s}v_i$ for some integers s and i and some $u_1, \dots, u_{2s} \in V(G)$. Note that some subset of M is a perfect matching of $G \setminus V(P \cup C)$. It follows that the paths $u_{2s}v_iv_{i+1}, v_{i+1}v_{i+2}v_{i+3}, \dots, v_{i-3}v_{i-2}v_{i-1}, v_{i-1}v_iu_{2s}$ are central in G and therefore directed (indices of vertices of C are taken modulo $2l+2$). Again it follows that C is oddly oriented. As every central circuit of G^u is oddly oriented, D is Pfaffian. □

For a factor-critical graph G and a vertex $v \in V(G)$ we define an auxiliary graph $G(v)$ with vertex set $N(v) = \{v' \in V(G) \mid v'v \in E(G)\}$ and let $v_1v_2 \in E(G(v))$ if and only if v_1vv_2 is a central path in G.

Lemma 5.5. *Let G be a 2-connected factor-critical graph. Then for every $v \in V(G)$ the graph $G(v)$ is connected.*

Proof. The proof is by induction on the number of ears in an ear decomposition (Theorem 5.1) of the graph G. The base case is trivial.

Let now $(C, P_1, \dots, P_k)$ be an ear decomposition of G and let $G' = C \cup P_1 \cup \dots \cup P_{k-1}$. By the induction hypothesis $G'(v)$ is connected for every $v \in V(G')$. Therefore $G(v)$ is connected for every $v \in V(G) - V(P_k)$. Let $P_k = v_0v_1 \dots v_{2l+1}$ for some integer $l \geq 0$. Note that $v_iv_{i+1}v_{i+2}$ is central in G for every $i \in \{0, \dots 2l-1\}$. Indeed, if i is even then a perfect matching of $G' \setminus v_0$ (which exists by Theorem 5.1) can be extended to a perfect matching of $G \setminus \{v_i, v_{i+1}, v_{i+2}\}$ and if i is odd then so can a perfect matching of $G' \setminus v_{2l+1}$. Therefore $G(v_i)$ is connected for every $i \in \{1, \dots 2l\}$.

The graph $G(v_0)$ is obtained from $G'(v_0)$ by the addition of the vertex v_1 and some edges. Therefore to show that $G(v_0)$ is connected it is sufficient to show that for some $w \in N(v_0)$ the path wv_0v_1 is central in G. Let M be a perfect matching of $G \setminus v_0$ and M' be a perfect matching of $G \setminus v_1$. There exists a component P of $G[M \cup M']$ such that P is an even path with one end in v_0 and the other end in v_1. Let $wv_0 \in E(P)$. Then $P \setminus \{w, v_0, v_1\}$ has a perfect matching and a subset of M is a perfect matching of $G \setminus V(P)$ and therefore wv_0v_1 is central in G. Similarly, $G(v_{2l+1})$ is connected. □

Theorem 5.6. *A factor-critical graph G is Pfaffian if and only if for every $v \in V(G)$ the graph $G(v)$ is bipartite.*

Proof. Let $v \in V(G)$ and let D be an orientation of G. For a vertex $w \in V(G(v))$ we say that w is *black* if $vw \in D$ and that w is *white* otherwise. If for some $w_1w_2 \in E(G(v))$ the vertices w_1 and w_2 have the same color then w_1vw_2 is a central path in G which is not directed. It follows that it is necessary for $G(v)$ to be bipartite for every $v \in V(G)$ for an orientation from Lemma 5.4 to exist.

We claim that the above condition is also sufficient. We prove our claim for 2-connected factor-critical graphs first. As in Lemma 5.5 we apply induction on the number of ears in an ear decomposition of graph G. The base case is immediate as odd cycles are Pfaffian.

Let now $(C, P_1, \ldots, P_k)$ be an ear decomposition of G, let $G' = C \cup P_1 \cup \ldots \cup P_{k-1}$ and let $P_k = v_0 v_1 \ldots v_{2l+1}$. By the induction hypothesis and Lemma 5.4 there exists an orientation D of G' such that every central path of length two is directed. By Lemma 5.5 there exists $w \in N(v_0)$ such that $wv_1 \in E(G(v_0))$. Extend D to an orientation D' of G by orienting the edges of P_k in such a way that $wv_0v_1 \ldots v_{2l+1}$ is a directed path. We claim that every central path of length two is directed in D'.

Suppose for some v, v', v'' the path $v'vv''$ is central in G, but not directed in D'. It follows that $v \in V(G')$. Suppose first $v \notin \{v_0, v_{2l+1}\}$. By Lemma 5.5 there exists a path between v' and v'' in $G'(v)$ and by the choice of D this path has to be even. It follows that $G(v)$ is not bipartite as $v'v'' \in G(v)$, in contradiction with our assumption. Note that by construction the same argument applies to $v = v_0$ (if $v = v_0$ and say $v'' = v_1$, then we apply the above argument to the pair w, v' instead).

It remains to consider $v = v_{2l+1}$. Since $v'vw$ is central there exists a perfect matching M of $G \setminus \{v, v', v''\}$. Let M' be a perfect matching of $G \setminus v$ and let P be a path with edges in $M \cup M'$ and ends in v' and v''. Let C be the cycle with $E(C) = E(P) \cup \{vv', vv''\}$. There must exist a subpath $t'tt''$ of C such that $t \neq v$ and $t'tt''$ is not directed. Note that C is central in G and therefore so is $t'tt''$. But we have already proved that for every $t \in V(G)$, $t \neq v_{2l+1}$ every central path of length two with the middle vertex in t is directed. This concludes the proof for 2-connected factor-critical graphs.

By Lemma 5.2 every block B of G is factor-critical and therefore we proved that there exists an orientation of B in which every length 2 central path is directed. Let D be an orientation of G constructed by combining such orientations for all blocks. It follows from Lemma 5.3 that every length 2 central path in G is directed. □

Theorem 5.6 provides a polynomial time recognition algorithm to decide whether a factor-critical graph is Pfaffian. Furthermore, the proof of Theorem 5.6 can be converted to an algorithm to find a Pfaffian orientation of G^u when it exists. Alternatively, one can use the algorithm of Vazirani and Yannakakis [17] that determines Pfaffian orientation of a Pfaffian graph in polynomial time.

6. Minimally non-Pfaffian factor-critical graphs

In this section we characterize non-Pfaffian factor-critical graphs in terms of forbidden central subgraphs. We will need a lemma about intersection of M-alternating paths from [13]. A path P is said to be *M-alternating*, if every internal vertex of P is incident with an edge of $E(P) \cap M$. We have to precede the statement of the lemma with a technical definition. Let G be a graph, let M be a matching in G, and let P and Q be two M-alternating paths in G. For the purpose of this definition let a *segment* be a maximal subpath of $P \cap Q$, and let an *arc* be a maximal subpath of Q with no internal vertex or edge in P. We say that P and Q *intersect transversally* if either they are vertex-disjoint, or there exist vertices $q_0, q_1, \ldots, q_7 \in V(Q)$ such that

(1) $q_0, q_1, \ldots, q_7$ occur on Q in the order listed, and q_0 and q_7 are the ends of Q,
(2) $q_2, q_1, q_3, q_4, q_6, q_5$ all belong to P and occur on P in the order listed,
(3) if $q_0 \in V(P)$, then $q_0 = q_1 = q_2 = q_3$, and otherwise $Q[q_0, q_1]$ is an arc,
(4) if $q_7 \in V(P)$, then $q_7 = q_6 = q_5 = q_4$, and otherwise $Q[q_6, q_7]$ is an arc,
(5) $Q[q_3, q_4]$ is a segment,
(6) either $q_1 = q_2 = q_3$, or q_1, q_2, q_3 are pairwise distinct, $Q[q_1, q_2]$ is a segment, $Q[q_2, q_3]$ is an arc and q_2 is not an end of P, and
(7) either $q_4 = q_5 = q_6$, or q_4, q_5, q_6 are pairwise distinct, $Q[q_5, q_6]$ is a segment, $Q[q_4, q_5]$ is an arc and q_5 is not an end of P.

The definition above is symmetric in P and Q. There are four cases of transversal intersection depending on the number of components of $P \cap Q$; the three cases when P and Q intersect are depicted in Figure 5. We are now ready to state the lemma from [13].

Lemma 6.1. *Let M be a matching in a graph G and let P_1 and P_2 be two M-alternating paths, where P_i has ends s_i and t_i. Assume that s_1, s_2, t_1 and t_2 have degree at most two in $P_1 \cup P_2$. Then there exist a matching M' saturating the same set of vertices as M and two M'-alternating paths Q_1 and Q_2 such that $M \triangle M' \subseteq E(P_1) \cup E(P_2)$, Q_i has ends s_i and t_i and Q_1 and Q_2 intersect transversally.*

Let G be a graph, let $k \geq 3$ be an odd integer, and let $v, w_1, w_2, \ldots w_k \in V(G)$ be distinct. Let $P_1, P_2, \ldots, P_k$, $Q_1, \ldots, Q_k$ be internally disjoint paths in G such that the following conditions are satisfied

- for $1 \leq i \leq k$ the path P_i is even and has ends v and w_i,

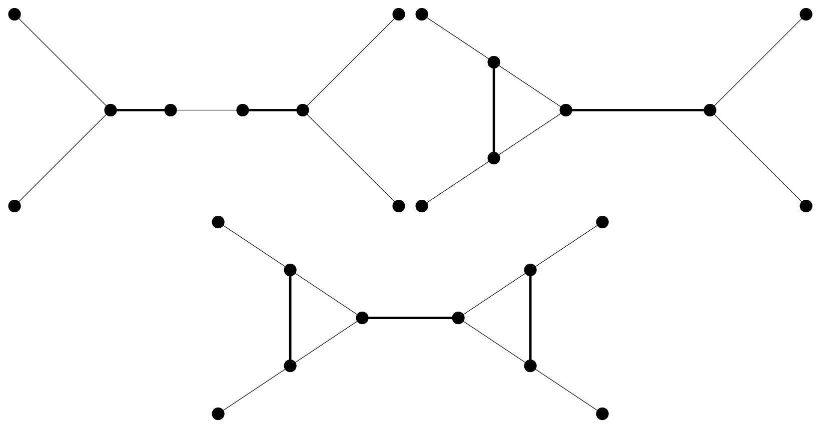

FIGURE 5. Three cases of transversal intersection.

- for $1 \leq i \leq k$ the path Q_i is odd and has ends w_i and w_{i+1}, where $w_{k+1} = w_1$ by convention, and
- $G = P_1 \cup \ldots \cup P_k \cup Q_1 \cup \ldots \cup Q_k$.

Then we say that G is a *k-flower*, v is the *hub* of G and the vertices of G adjacent to v are the *spokes* of G. If a graph H is obtained from a 3-flower G by contracting the unique odd cycle not containing the hub, then we say that H is a *pseudoflower*. The *hub* and the *spokes* of H are the images of the hub and the spokes of G under this contraction. We will show that k-flowers and pseudoflowers are non-Pfaffian, and that every non-Pfaffian factor-critical graph contains a k-flower or a pseudoflower as a central subgraph.

Lemma 6.2. *For every odd integer $k \geq 3$ every k-flower G is non-Pfaffian. Every pseudoflower G is non-Pfaffian.*

Proof. Let v be the hub of G, and let H be obtained from G^u by deleting all edges uw, where w is not a spoke. If G is a pseudoflower, then H has a matching minor isomorphic to $K_{3,3}$, and if G is a $(2t+1)$-flower, then H is isomorphic to $G(t, 0)$. Thus G is not Pfaffian by Theorem 1.1 and Theorem 3.2. □

It is not hard to see that if one deletes an edge from a flower or a pseudoflower then the resulting graph is Pfaffian.

Theorem 6.3. *Let G be a factor-critical graph, let $v \in V(G)$ and let C be an induced odd cycle in $G(v)$ with $|C| = k$. Then there exists a k-flower or a pseudoflower F such that F is a central subgraph of G, v is the hub of F and $V(C)$ is the set of spokes of F.*

Proof. Let $C = v_1 v_2 \ldots v_k v_1$.

We start by considering the case $k = 3$. Let M_i be a perfect matching of $G \setminus \{v, v_j, v_k\}$, where $\{i, j, k\} = \{1, 2, 3\}$. Note that $M_2 \triangle M_3$ is the union of cycles and a path with ends v_2 and v_3. Denote this path by P_1. Let P_2 be defined analogously. By Lemma 6.1 applied to M_3, P_1 and P_2 we may assume that P_1 and P_2 intersect transversally. Then the graph $F = G[E(P_1 \cup P_2) \cup \{vv_1, vv_2, vv_3\}]$ is a 3-flower (if $P_1 \cup P_2$ induces a cycle) or a pseudoflower (if $P_1 \cap P_2$ is a path). Moreover, F is central as M_3 induces a perfect matching in $G \setminus V(F)$.

Now assume $k > 3$. We need another technical definition similar to the one of k-flower. For $i \in \{2, \dots, l-1\}$ let P_i be an even path with ends v and w_i, and for $i = \{2, \dots, l-2\}$ let Q_i be an odd path with ends w_i and w_{i+1}. Let P_1 be an odd path with ends v and w_2 and let P_l be an odd path with ends v and w_{l-1}. If the paths $P_1, P_2, \dots P_l, Q_1, \dots, Q_{l-1}$ are pairwise internally vertex-disjoint, $vv_i \in E(P_i)$ for all $i = \{1, \dots, l\}$ and the graph $B = G[E(P_1 \cup P_2 \cup \dots P_l \cup Q_2 \cup \dots \cup Q_{l-2})]$ is a central subgraph of G then we say that B is an *l-blossom*.

Claim 1. *For every integer $4 \leq l \leq k-1$ there exists an l-blossom B.*

Proof. By induction on l.

We start with the base case $l = 4$. For $i \in \{1, 2, 3\}$ let M_i be a perfect matching of $G \setminus \{v, v_i, v_{i+1}\}$. For $i \in \{1, 3\}$ let R_i be the unique M_iM_2-alternating path; then R_1 has ends v_1 and v_3, while R_3 has ends v_2 and v_4. By Lemma 5.4 applied to M_2, R_1 and R_3 we may assume that R_1 and R_3 intersect transversally. We distinguish between the types of transversal intersection as follows:

(1) R_1 and R_3 are disjoint,
(2) $R_1 \cup R_3$ is connected and acyclic,
(3) $R_1 \cup R_3$ is connected and contains exactly one cycle,
(4) $R_1 \cup R_3$ is connected and contains exactly two cycles.

Let $B = G[E(R_1 \cup R_3) \cup \{vv_1, vv_2, vv_3, vv_4\}]$. Note that B is central in G as M_2 induces a perfect matching of $G \setminus V(B)$. Since C is induced, $G \setminus \{v, v_i, v_j\}$ has no perfect matching whenever $1 \leq i, j \leq 4$ and $|i - j| > 1$. Thus $B \setminus \{v, v_i, v_j\}$ has no perfect matching for those values of i, j. It follows that (2) holds and that B is a 4-blossom.

For the induction step, let $5 \leq l \leq k-1$ and let B be an $(l-1)$-blossom with notation as above. We proceed to construct an l-blossom. Let M be a perfect matching of $G \setminus V(B)$ and let M_l be a perfect matching of $G \setminus \{v, v_{l-1}, v_l\}$. Let R be the unique M_lM-alternating path with one end in v_l and the other end $w \in V(B) - \{v_{l-1}, v\}$. We claim that $w \in V(P_{l-1})$ as otherwise C is not induced. If $w \in V(P_i)$ for some $1 \leq i < l-1$ then either $v_iv_l \in E(G(v))$ or $v_{i+1}v_l \in E(G(v))$; other cases are analogous. Let $P_l = R, w_{l-1} = w$, replace P_{l-1} by $P_{l-1}[v, w]$ and let $Q_{l-1} = P_{l-1}[w, w_{l-2}]$.

Note that P_{l-1} is even and Q_{l-1} is odd as otherwise $B \cup R \setminus \{v, v_{l-2}, v_l\}$ has a perfect matching and $v_{l-2}v_l \in E(G(v))$. Therefore $G[E(B \cup R)]$ is an l-blossom. □

It now remains to construct a k-flower from a $(k-1)$-blossom B. Let M be a perfect matching of $G \setminus V(B)$. Let $B' = B \setminus (V(P_1) - \{w_2\})$, let $M_{B'} \supseteq M$ be a perfect matching of $G \setminus V(B')$ and let M' be a perfect matching of $G \setminus \{v, v_{k-1}, v_k\}$. Let R' be the $M'M_{B'}$-alternating path with ends v_k and $w' \in V(B')$. By the argument from Claim 1 we have $w' \in V(P_{k-1})$ and $P_{k-1}[v, w']$ is an odd path. Suppose now that $P_1 \cap R' \neq \emptyset$. Let $w \in V(P_1 \cap R')$ be chosen to minimize $R'[w, w']$. By examining $B \cup R'[w, w']$ we can conclude that $v_{k-1}v_1 \in E(G(v))$, or $v_{k-1}v_2 \in E(G(v))$, or $v_{k-2}v_1 \in E(G(v))$, in contradiction with the choice of C.

Similarly let $B'' = B \setminus (V(P_{k-1}) - \{w_{k-2}\})$, let $M_{B''} \supseteq M$ be a perfect matching of $G - V(B'')$ and let M'' be a perfect matching of $G - \{v, v_1, v_k\}$. Then there exists $M''M_{B''}$-alternating path R'' with ends v_k and $w'' \in V(P_1)$,such that $P_1[v, w'']$ is odd and R'' is otherwise disjoint from B. Note that R' and R'' are both M-alternating and we can apply Lemma 5.4 to M, R' and R''. It is easy to see that $R' \cup R''$ is acyclic by the choice of C and therefore $B \cup R' \cup R''$ constitutes a k-flower. □

Corollary 6.4. *A factor-critical graph G is non-Pfaffian if and only if G contains a central subgraph that is a pseudoflower or a k-flower for some integer $k \geq 3$.*

Proof. Note that every central subgraph of a Pfaffian graph is Pfaffian. Therefore, if G contains a central subgraph F that is a pseudoflower or a k-flower, then G is non-Pfaffian by Lemma 6.2.

If G is non-Pfaffian then by Theorem 5.6 there exists $v \in V(G)$ such that the auxiliary graph $G(v)$ is non-bipartite. By Theorem 6.3 there exists a k-flower or a pseudoflower F such that F is a central subgraph of G, and v is the hub of F. □

Note that the proof of Lemma 5.4 is algorithmic and so are the proofs in this section; therefore in a non-Pfaffian factor-critical graph it is possible to find a k-flower or a pseudoflower in polynomial time.

7. Concluding remarks

A *cut* in a graph G is a set $\delta(S)$ of all edges joining vertices of S to vertices of $V(G) - S$ for some non-empty $S \subsetneq V(G)$. We say that a cut is *trivial* if S or $V(G) - S$ contains only one vertex. We say that an odd cut C in a graph G is *tight* if every perfect matching of G contains exactly one edge in it.

The tight cut decomposition procedure of Kotzig, and Lovász and Plummer [9] can be used to reduce most of the problems regarding perfect matchings to matching covered graphs with no non-trivial tight cuts. In particular, it suffices to characterize Pfaffian graphs with no non-trivial tight cut. There are two such classes of graphs. A *brick* is a 3-connected bicritical graph, where a graph G is *bicritical* if $G \setminus \{u,v\}$ has a perfect matching for every two distinct vertices $u, v \in V(G)$. A *brace* is a connected bipartite graph such that every matching of size at most two is contained in a perfect matching. Edmonds, Lovász and Pulleyblank [1] and Lovász [10] proved that a matching-covered graph has no non-trivial tight cuts if and only if it is either a brick or a brace.

Pfaffian bipartite graphs are well understood. Therefore it suffices to characterize Pfaffian bricks. While the problem of enumerating near-perfect matchings provides an independent motivation for our study of Pfaffian factor-critical graphs, one can consider this study as an attempt to approach and gain intuition about the substantially more difficult problem of characterizing Pfaffian bricks. Clearly, a graph G is 2-connected and factor-critical if and only if G^u is a brick. A vertex u of a graph G is said to be *universal* if $uv \in E(G)$ for every $v \in V(G) - \{u\}$. One can consider the results of Sections 5 and 6 as characterizations of Pfaffian bricks with a universal vertex.

Already in this special case minimally non-Pfaffian graphs constitute an infinite family (in fact, Lemma 6.2 offers a glimpse at the relation between this family and the family considered in Section 3). The exact description of this family is obtained in Section 6. This fact seems to offer hope that such a description, while much harder to obtain, might be possible for general Pfaffian bricks.

A completely different approach to characterizing Pfaffian graphs is by means of a structural theorem. For bipartite graphs such a theorem was obtained independently by McCuaig [11], and Robertson, Seymour and Thomas [15]. No such theorem is known for general non-bipartite graphs, but we hope to shed some light on this question in a forthcoming paper [12].

We finish the paper by further specializing our area of interest. First, we give a precise structural description of Pfaffian bricks with two universal vertices.

Theorem 7.1. *Let G be a brick and let $u_1, u_2 \in V(G)$ be universal. Then G is Pfaffian if and only if $G' = G \setminus \{u_1, u_2\}$ is bipartite and has a unique perfect matching.*

Proof. Let M be a perfect matching of G'; it exists as G is bicritical. Suppose G' contains an odd cycle. For an odd cycle C in G' let M_c be the set of edges of M that are incident to a vertex of C. Choose C with $|M_c|$ minimal. Then no edge of M forms a chord of C. Let $V(M_c) - V(C) = \{v_1, v_2, \dots v_{2k+1}\}$. Let $F = C \cup M_c \cup (\bigcup_{i=1}^{2k+1} u_1 v_i)$. Then F is a $(2k+1)$-flower and is central in $G \backslash u_2$ unless $k = 0$, in which case $G[\{u_1\} \cup V(M_c) \cup V(C)]$ contains a spanning pseudoflower. It follows from Lemma 6.2 that if G' is non-bipartite then G is non-Pfaffian. Suppose G' has two perfect matchings. Then their symmetric difference is a union of central cycles. Let C_0 be a central cycle in G, and let v_1, v_2 be two vertices even distance apart in C_0. Then $C_0 + u_1v_1 + u_1v_2$ is a central pseudoflower in $G \setminus u_2$ and it again follows that G is non-Pfaffian.

It remains to show that if G' is bipartite and has a unique perfect matching M then G is Pfaffian. Let (A, B) be a bipartition of G'. We construct the Pfaffian orientation D of G as follows: direct the edges of M from A to B, direct all other edges of G' from B to A, direct all edges from u_1 and u_2 to A, from B to u_1 and u_2, and direct the edge u_1u_2 from u_1 to u_2.

Let $M' = M \cup \{u_1u_2\}$. We claim that every M'-alternating cycle C is oddly oriented in D. Note that $u_1u_2 \in E(C)$, as otherwise C is a central cycle in G'. If an edge e of C incident to u_1, but not to u_2, has an end in A then all the edges of C except u_1u_2 are oriented in the same direction along C, and therefore C is oddly oriented. The case when e has an end in B is similar. Thus our claim holds, and the orientation D is Pfaffian, as every perfect matching of G has the same sign as M' in D. □

Finally, let us give a characterization of Pfaffian graphs in a certain class that includes $G(k, \emptyset)$ for every k. Let H be a graph, let V be a set of vertices disjoint from $V(H)$ and let $f : V \to V(H)$ be one-to-one. Define $G(H)$ as follows: $V(G(H)) = V(H) \cup V \cup \{w_1, w_2\}$, $E(G(H)) = E(H) \cup (\bigcup_{v \in V} \{vf(v), vw_1, vw_2\})$.

Lemma 7.2. *The graph $G(H)$ is Pfaffian if and only if the graph H is bipartite.*

Proof. If H' is a subgraph of H then $G(H')$ is isomorphic to a central subgraph of $G(H)$. If H is not bipartite then $G(H)$ has $G(C_{2k+1}) \simeq G(k, \emptyset)$ as a subgraph and is therefore non-Pfaffian. If H is bipartite then H is a subgraph of a graph $G^+(H)$ obtained from $G(H)[V(H) \cup V]$ by adding two universal vertices. The graph $G(H)[V(H) \cup V]$ is bipartite and has a unique prefect matching. Therefore by Theorem 7.1 $G^+(H)$ is Pfaffian and therefore so is $G(H)$. □

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