A survey of Pfaffian orientations of graphs

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Abstract. An orientation of a graph G is Pfaffian if every even cycle C such that $G\backslash V(C)$ has a perfect matching has an odd number of edges directed in either direction of the cycle. The significance of Pfaffian orientations is that if a graph has one, then the number of perfect matchings (a.k.a. the dimer problem) can be computed in polynomial time.

The question of which bipartite graphs have Pfaffian orientations is equivalent to many other problems of interest, such as a permanent problem of Pólya, the even directed cycle problem, or the sign-nonsingular matrix problem for square matrices. These problems are now reasonably well-understood. On the other hand, it is not known how to efficiently test if a general graph is Pfaffian, but there are some interesting connections with crossing numbers and signs of edge-colorings of regular graphs.

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1. Introduction

All *graphs* in this paper are finite, do not have loops or multiple edges and are undirected. *Directed graphs*, or *digraphs*, do not have loops or multiple edges, but may have two edges between the same pair of vertices, one in each direction. Most of our terminology is standard and can be found in many textbooks, such as [4], [10], [65]. In particular, *cycles* and *paths* have no repeated vertices. A subgraph H of a graph G is called *central* if $G\backslash V(H)$ has a perfect matching (we use $\backslash$ for deletion). An even cycle C in a directed graph D is called *oddly oriented* if for either choice of direction of traversal around C, the number of edges of C directed in the direction of traversal is odd. Since C is even, this is clearly independent of the initial choice of direction of traversal. Finally, an orientation D of (the edges of) a graph G is *Pfaffian* if every even central cycle of G is oddly oriented in D. We say that a graph G is *Pfaffian* if it has a Pfaffian orientation.

The significance of Pfaffian orientations stems from the fact that if a graph G has one, then the number of perfect matchings of G (as well as other related problems) can be computed in polynomial time. We survey this in Section 2. The following

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is a classical theorem of Kasteleyn [23]. A special case is implicit in the work of Fisher [16] and Temperley and Fisher [55]. Different proofs may be found in [25], [26], [32], [38].

Theorem 1.1. *Every planar graph is Pfaffian.*

The smallest non-Pfaffian graph is the complete bipartite graph $K_{3,3}$. This paper is centered around the question of which graphs are Pfaffian. For bipartite graphs this is equivalent to many other problems of interest, and is by now reasonably well-understood. We list several such problems in Section 3, including a question of Pólya from 1913 whether the permanent of a square matrix can be calculated by a reduction to a determinant of a related matrix, the even directed cycle problem for digraphs, and the sign-nonsingular matrix problem. In Section 4 we discuss two characterizations of bipartite Pfaffian graphs. The first is in terms of excluded obstructions; it turns out that for bipartite graphs $K_{3,3}$ is the only obstruction with respect to the "matching minor" partial order, defined later. This is an analogue of the graph minor relation and is well-suited for problems involving perfect matchings. Unfortunately, it no longer has many of the nice properties of the usual minor order. The second characterization is structural and describes the structure of all bipartite Pfaffian graphs. It turns out those graphs and only those graphs can be built from planar graphs and one sporadic nonplanar graph by certain composition operations. This characterization implies a polynomial-time algorithm to decide whether a bipartite graph is Pfaffian, and hence solves all the problems listed in Section 3. Applications of the structure theorem are discussed in Section 5.

We then turn to general graphs. In Section 6 we review a matching decomposition procedure of Lovász and Plummer that decomposes every graph into "bricks" and "braces". The decomposition has the property that a graph is Pfaffian if and only if all its constituent bricks and braces are Pfaffian. Furthermore, braces are bipartite, and hence whether they are Pfaffian can be decided using the algorithm of Section 4. Thus in order to test whether an input graph is Pfaffian it suffices to design an algorithm for bricks. Motivated by this we present a recent theorem that describes how to construct an arbitrary brick, and later we discuss various examples and results that were obtained using this theorem.

In the next section we talk about results of Norine that relate Pfaffian graphs and crossing numbers. The starting point here is Theorem 7.1 that characterizes Pfaffian graphs in terms of drawings in the plane. Norine then generalized it to T-joins, whereby the generalization implies several well-known results about crossing numbers, and in a different direction proved an analogue for 4-Pfaffian graphs and drawings in the torus. The latter suggests a general conjecture that is still open.

In Section 8 we discuss the relationship between signs of edge-colorings (in the sense of Penrose [46]) and Pfaffian orientations. We mention a proof of a conjecture of Goddyn that in a k-regular Pfaffian graph all k-edge-colorings have the same sign, which holds more generally for graphs that admit a "Pfaffian labeling." We present a partial converse of this, and then describe two characterizations of graphs that admit a

Pfaffian labeling. The above research led Norine and the author to make the following conjecture [44].

Conjecture 1.2. Every 2-connected 3-regular Pfaffian graph is 3-edge-colorable.

Let us recall that by Tait's result [54] (see also [65]) the Four-Color Theorem is equivalent to the statement that every 2-connected 3-regular planar graph is 3-edge-colorable. Thus, if true, Conjecture 1.2 would imply the Four-Color Theorem by Theorem 1.1.

In the last section we discuss the prospects for characterizing general Pfaffian graphs, either structurally or by means of excluded matching minors.

2. Pfaffian orientations and counting perfect matchings

Pfaffian orientations were invented by the physicists M. E. Fisher, P. W. Kasteleyn, and H. N. V. Temperley as a tool for enumerating the number of perfect matchings in a graph (or, in physics terminology, to solve the dimer problem). Let us start by explaining their approach. Let $A = (a_{ij})$ be a skew symmetric $n \times n$ matrix; that is $a_{ij} = -a_{ji}$. For each partition $\pi = \{\{i_1, j_1\}, \{i_2, j_2\}, \dots, \{i_k, j_k\}\}$ of the set $\{1, 2, \dots, n\}$ into unordered pairs ("partition into pairs") we define the quantity

$$\sigma_\pi = \operatorname{sgn} \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2k-1 & 2k \\ i_1 & j_1 & i_2 & j_2 & \dots & i_k & j_k \end{pmatrix} a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_k j_k}, \tag{1}$$

where sgn denotes the sign of the indicated permutation. Clearly, there is no partition into pairs if n is odd. The *Pfaffian* of A is defined by $\mathrm{Pf}(A) = \sum \sigma_\pi$, where the summation is over all partitions of $\{1, 2, \dots, n\}$ into pairs. Since A is skew symmetric the value of σ_π does not depend on the order of blocks of π or on the order in which the members of a block are listed, and hence $\mathrm{Pf}(A)$ is well-defined. We will need the following lemma from linear algebra [23], [37].

Lemma 2.1. *If A is a skew symmetric matrix, then* $\det A = (\mathrm{Pf}(A))^2$.

Now let G be a graph with vertex-set $\{1, 2, \dots, n\}$, and let D be an orientation of (the edges of) G. To the orientation D there corresponds a skew adjacency matrix $A = (a_{ij})$ of G defined by saying that $a_{ij} = 0$ if i is not adjacent to j, and otherwise $a_{ij} = 1$ if the edge ij is directed in D from i to j and $a_{ij} = -1$ if the edge ij is directed in D from j to i. If π is a partition of $\{1, 2, \dots, n\}$ into pairs, then $\sigma_\pi \neq 0$ if and only if each pair in π is an edge of G, or, in other words, π is a perfect matching of G. Thus the summation in the definition of $\mathrm{Pf}(A)$ might as well be restricted to perfect matchings of G. We define $\mathrm{sgn}_D(M)$, the *sign of a perfect matching M of D*, as σ_M, or, equivalently, by

$$\mathrm{sgn}_D(M) = \operatorname{sgn} \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2k-1 & 2k \\ i_1 & j_1 & i_2 & j_2 & \dots & i_k & j_k \end{pmatrix}, \tag{2}$$

where the edges of M are listed as $i_1 j_1, i_2 j_2, \ldots, i_k j_k$ in such a way that $i_t j_t$ is directed from i_t to j_t in D. It is not hard to see that D is a Pfaffian orientation of G if and only if $\mathrm{sgn}_D(M)$ does not depend on M. If that is the case, then $|\mathrm{Pf}(A)|$ is equal to the number of perfect matchings of G, and by Lemma 2.1 the number of perfect matchings of G can be computed efficiently.

This is significant, because Valiant [63] proved that counting the number of perfect matchings in general graphs (even in bipartite graphs) is #P-complete, and therefore is unlikely to be polynomial-time solvable. Furthermore, Theorem 1.1 guarantees that there is an interesting and useful class of graphs for which this technique can be applied.

The dimer problem of statistical mechanics is concerned with the properties of a system of diatomic molecules, or dimers, adsorbed on the surface of a crystal. Usually it is assumed that the adsorption points form the vertices of a lattice graph, such as the 2-dimensional grid. A crucial problem in the calculation of the thermodynamic properties of such a system of dimers is that of enumerating all ways in which a given number of dimers can be arranged on the lattice without overlapping each other. In the related monomer-dimer model some sites may be left unoccupied, but in the dimer model it is assumed that the dimers cover all the vertices of the graph; in other words, they form a perfect matching. Kasteleyn [21], [22], [23], Fisher [16] and Temperley and Fisher [55] used the method described in this section to solve the 2-dimensional dimer problem. The method is more general in the sense that it allows the computation of the dimer partition function, and that, in turn, can be used to solve the 2-dimensional Ising problem [23]. Let us remark that the 3-dimensional dimer problem remains open.

3. Some equivalent problems

Vazirani and Yannakakis [64] used a deep theorem of Lovász [31] to show the following.

Theorem 3.1. *The decision problems "Is a given orientation of a graph Pfaffian" and "Is an input graph Pfaffian" are polynomial-time equivalent.*

This is reasonably easy for bipartite graphs. There does not seem to be an elementary proof for general graphs, but the theorem can be easily deduced from the results discussed in Section 6.

Computing the permanent of a matrix seems to be of a different computational complexity from computing the determinant. While the determinant can be calculated using Gaussian elimination, no efficient algorithm for computing the permanent is known, and, in fact, none is believed to exist. More precisely, Valiant [63] has shown that computing the permanent is #P-complete even when restricted to 0-1 matrices.

It is therefore reasonable to ask if perhaps computing the permanent can be somehow reduced to computing the determinant of a related matrix. In particular, the following question was asked by Pólya [47] in 1913. If A is a 0-1 square matrix, does there exist a matrix B obtained from A by changing some of the 1's to -1's in such a way that the permanent of A equals the determinant of B? For the purpose of this paper let us say that B (when it exists) is a *Pólya matrix* for A.

Let G be a bipartite graph with bipartition (X, Y). The *bipartite adjacency matrix of G* has rows indexed by X, columns indexed by Y, and the entry in row x and column y is 1 or 0 depending on whether x is adjacent to y or not. Vazirani and Yannakakis [64] proved the following.

Theorem 3.2. *Let G be a bipartite graph, and let A be its bipartite adjacency matrix. Then A has a Pólya matrix if and only if G has a Pfaffian orientation.*

Let us turn to directed graphs now. A digraph D is *even* if for every weight function $w\colon E(D) \to \{0, 1\}$ there exists a cycle in D of even total weight. It was shown in [53] and is not difficult to see that testing evenness is polynomial-time equivalent to testing whether a digraph has an even directed cycle. (This is equivalent to Theorem 3.1 for bipartite graphs.) Let G be a bipartite graph with bipartition (A, B), and let M be a perfect matching in G. Let $D = D(G, M)$ be obtained from G by directing every edge from A to B, and contracting every edge of M. Little [27] has shown the following.

Lemma 3.3. *Let G be a bipartite graph, and let M be a perfect matching in G. Then G has a Pfaffian orientation if and only if $D(G, M)$ is not even.*

We say that two $n \times m$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ have the same *sign-pattern* if for all pairs of indices i, j the entries a_{ij} and b_{ij} have the same sign; that is, they are both strictly positive, or they are both strictly negative, or they are both zero. A square matrix A is *sign-nonsingular* if every real matrix with the same sign pattern is nonsingular.

In economic analysis one may not know the exact quantitative relationships between different variables, but there may be some qualitative information such as that one quantity rises if and only if another does. For instance, it is generally agreed that the supply of a particular commodity increases as the price increases, even though the exact dependence may vary. Thus we may want to deduce qualitative information about the solution to a linear system $A\boldsymbol{x} = \boldsymbol{b}$ from the knowledge of the sign-patterns of the matrix A and vector $\boldsymbol{b}$. That motivates the following definition. We say that the linear system $A\boldsymbol{x} = \boldsymbol{b}$ is *sign-solvable* if for every real matrix B with the same sign-pattern as A and every vector $\boldsymbol{c}$ with the same sign-pattern as $\boldsymbol{b}$ the system $B\boldsymbol{y} = \boldsymbol{c}$ has a unique solution $\boldsymbol{y}$, and its sign-pattern does not depend on the choice of B and $\boldsymbol{c}$. The study of sign-solvability was first proposed by Samuelson [51].

It follows from standard linear algebra that sign-solvability can be decided efficiently if and only if sign-nonsingularity can. But for square matrices the latter is equivalent to testing whether a given orientation of a bipartite graph is Pfaffian. To state the result, let D be a bipartite digraph with bipartition (X, Y). The *directed bipartite adjacency matrix of D* has rows indexed by X, columns indexed by Y, and the entry in row x and column y is 1, -1 or 0 depending on whether D has an edge directed from x to y, or D has an edge directed from y to x, or x and y are not adjacent in D. By Theorem 3.1 the following result implies that testing sign-solvability is polynomial-time equivalent to testing whether a bipartite graph is Pfaffian.

Theorem 3.4. *Let D be a directed bipartite graph with a perfect matching, and let A be its directed bipartite adjacency matrix. Then A is sign-nonsingular if and only if D is a Pfaffian orientation of its underlying undirected graph.*

The next problem is about hypergraph coloring. A *hypergraph* H is a pair $(V(H), E(H))$, where $V(H)$ is a finite set and $E(H)$ is a collection of distinct nonempty subsets of $V(H)$. We say that H is 2-*colorable* if $V(H)$ can be colored using two colors in such a way that every edge includes vertices of both colors. We say that H is *minimally non*-2-*colorable* if H is not 2-colorable, has no isolated vertices, and the deletion of any member of $E(H)$ results in a 2-colorable hypergraph. Seymour [52] proved the following.

Theorem 3.5. *Let H be a hypergraph with no isolated vertices and $|E(H)| = |V(H)|$, let D be the digraph with bipartition $(V(H), E(H))$ defined by saying that D has an edge directed from $v \in V(H)$ to $E \in E(H)$ if and only if $v \in E$, and let G be the underlying undirected graph of D. Then H is minimally non-2-colorable if and only if G is connected, every edge of G belongs to a perfect matching of G and D is a Pfaffian orientation of G.*

Our last problem is about the polytope of even permutation matrices. The convex hull of permutation matrices has been characterized by Birkhoff [3] as precisely the set of doubly stochastic matrices. It is an open problem to characterize the convex hull of even permutation matrices. More precisely, it is not known if there exists a polynomial-time algorithm to test whether a given $n \times n$ matrix belongs to this polytope. By a fundamental result of Grötschel, Lovász and Schrijver [19] this problem is solvable in polynomial time if there exists a polynomial-time algorithm for the optimization problem: Given a fixed $n \times n$ matrix M, find the maximum of $M \cdot X$ over all even permutation matrices X, where "$\cdot$" denotes the dot product in $\mathbb{R}^{n^2}$ and both matrices are regarded as vectors of length n^2.

A special case of the above optimization problem when A is a 0-1 matrix and we want to determine if the maximum is n can be reformulated as follows. Let G be a bipartite graph with bipartition (A, B), and let D be the orientation of G defined by orienting every edge from A to B. The problem is: "Decide if G has a perfect matching M such that $\mathrm{sgn}_D(M) = 1$." By Theorem 3.1 this is polynomial-time equivalent to deciding whether a bipartite graph has a Pfaffian orientation.

4. Characterizing bipartite Pfaffian graphs

We have seen in the previous section that characterizing bipartite Pfaffian graphs is of interest. In this section we discuss two such characterizations and a recognition algorithm. We begin with an elegant theorem of Little [27]. Let H be a graph, and let v be a vertex of H of degree two. By *bicontracting* v we mean contracting both edges incident with v and deleting the resulting loops and parallel edges. A graph G is a *matching minor* of a graph H if G can be obtained from a central subgraph of H by repeatedly bicontracting vertices of degree two. It is fairly easy to see that a matching minor of a Pfaffian graph is Pfaffian.

Theorem 4.1. *A bipartite graph admits a Pfaffian orientation if and only if it has no matching minor isomorphic to* $K_{3,3}$.

By Lemma 3.3 the above implies a characterization of even digraphs. Seymour and Thomassen obtained such characterization from first principles in [53]. Interestingly, the latter involves infinitely many excluded minors, rather than one.

Unfortunately, Theorem 4.1 does not seem to imply a polynomial-time algorithm to test whether a bipartite graph is Pfaffian, the difficulty being that it is not clear how to efficiently test for the presence of a matching minor isomorphic to $K_{3,3}$. The next result gives a structural description of bipartite Pfaffian graphs, and can be used to derive a polynomial-time recognition algorithm. We need some definitions first.

Let G_0 be a graph, let C be a central cycle of G_0 of length four, and let G_1, G_2, G_3 be three subgraphs of G_0 such that $G_1 \cup G_2 \cup G_3 = G_0$, and for distinct integers $i, j \in \{1, 2, 3\}$, $G_i \cap G_j = C$ and $V(G_i) - V(C) \neq \emptyset$. Let G be obtained from G_0 by deleting some (possibly none) of the edges of C. In these circumstances we say that G is a *trisum* of G_1, G_2 and G_3. The *Heawood graph* is the bipartite graph associated with the incidence matrix of the Fano plane (see Figure 1).

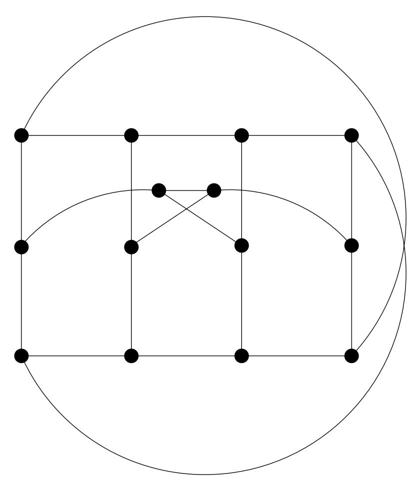

Figure 1. The Heawood graph.

A graph G is *k-extendable*, where $k \geq 0$ is an integer, if every matching of size at most k can be extended to a perfect matching. A connected 2-extendable bipartite graph is called a *brace*. It is easy to see (and will be outlined in Section 6) that the problem of finding Pfaffian orientations of bipartite graphs can be reduced to braces. The following was shown in [35] and, independently, in [50].

Theorem 4.2. *A brace has a Pfaffian orientation if and only if either it is isomorphic to the Heawood graph, or it can be obtained from planar braces by repeated application of the trisum operation.*

Let us turn to testing whether a bipartite graph is Pfaffian. We wish to apply Theorem 4.2, and for that the following result [50, Theorem 8.3] is very helpful.

Theorem 4.3. *Let G be a brace that has a Pfaffian orientation, and let G be a trisum of G_1, G_2 and G_3. Then G_1, G_2 and G_3 have a Pfaffian orientation.*

A polynomial-time algorithm now follows easily. Given a bipartite graph G we first decompose it into braces (more on that in Section 6), and apply the algorithm recursively to each brace in the decomposition. Thus we may assume that G is a brace. Now we test if G has a set $X \subseteq V(G)$ of size four such that $G \backslash X$ has at least three components. If it does, then G can be expressed as a trisum of three smaller graphs, and by Theorem 4.3 we may apply the algorithm recursively to each of the three smaller graphs. On the other hand, if G has no set X as above, then by Theorem 4.2 G is Pfaffian if and only if it is planar or isomorphic to the Heawood graph. It is clear that this is a polynomial-time algorithm. In [50] it is shown how to implement it to run in time $O(|V(G)|^3)$. By using more modern algorithmic results the running time can be reduced to $O(|V(G)|^2)$.

5. Applications of the characterization of bipartite Pfaffian graphs

As a corollary of Theorem 4.2 we get the following extremal result.

Corollary 5.1. *No brace with $n \geq 3$ vertices and more than $2n - 4$ edges has a Pfaffian orientation.*

Proof. Every planar bipartite graph on $n \geq 3$ vertices has at most $2n - 4$ edges. The result follows from Theorem 4.2 by induction. □

Since every digraph is isomorphic to $D(G, M)$ for some G and M, Theorem 4.2 gives a characterization of even directed graphs, using Lemma 3.3. Let us state the characterization explicitly, but first let us point out a relation between extendability and strong connectivity. A digraph D is *strongly connected* if for every two vertices u and v it has a directed path from u to v. It is *strongly k-connected*, where $k \geq 1$ is an integer, if for every set $X \subseteq V(D)$ of size less than k, the digraph $D \backslash X$ is strongly connected. The following is straightforward.

Lemma 5.2. *Let G be a connected bipartite graph, let M be a perfect matching in G, and let $k \geq 1$ be an integer. Then G is k-extendable if and only if $D(G, M)$ is strongly k-connected.*

Let D be a digraph, and let (X, Y) be a partition of $V(G)$ into two nonempty sets in such a way that no edge of G has tail in X and head in Y. Let $D_1 = D \backslash Y$ and $D_2 = D \backslash X$. We say that D is a 0-*sum* of D_1 and D_2. Now let $v \in V(D)$, and let (X, Y) be a partition of $V(D) - \{v\}$ into two nonempty sets such that no edge of D has tail in X and head in Y. Let D_1 be obtained from D by deleting all edges with both ends in $Y \cup \{v\}$ and identifying all vertices of $Y \cup \{v\}$, and let D_2 be obtained by deleting all edges with both ends in $X \cup \{v\}$ and identifying all vertices of $X \cup \{v\}$. We say that D is a 1-*sum* of D_1 and D_2. Let D_0 be a directed graph, let $u, v \in V(D_0)$, and let $uv, vu \in E(D_0)$. Let D_1 and D_2 be such that $D_1 \cup D_2 = D_0$, $V(D_1) \cap V(D_2) = \{u, v\}$, $V(D_1) - V(D_2) \neq \emptyset \neq V(D_2) - V(D_1)$ and $E(D_1) \cap E(D_2) = \{uv, vu\}$. Let D be obtained from D_0 by deleting some (possibly neither) of the edges uv, vu. We say that D is a 2-*sum* of D_1 and D_2. Now let D_0 be a directed graph, let $u, v, w \in V(D_0)$, let $uv, wv, wu \in E(D_0)$, and assume that D_0 has a directed cycle containing the edge wv, but not the vertex u. Let D_1 and D_2' be such that $D_1 \cup D_2' = D_0$, $V(D_1) \cap V(D_2') = \{u, v, w\}$, $V(D_1) - V(D_2') \neq \emptyset \neq V(D_2') - V(D_1)$ and $E(D_1) \cap E(D_2') = \{uv, wv, wu\}$, let D_2' have no edge with tail v, and no edge with head w. Let D be obtained from D_0 by deleting some (possibly none) of the edges uv, wv, wu, and let D_2 be obtained from D_2' by contracting the edge wv. We say that D is a 3-*sum* of D_1 and D_2. Finally let D_0 be a directed graph, let $x, y, u, v \in V(D_0)$, let $xy, xv, uy, uv \in E(D_0)$, and assume that D_0 has a directed cycle containing precisely two of the edges xy, xv, uy, uv. Let D_1 and D_2' be such that $D_1 \cup D_2' = D_0$, $V(D_1) \cap V(D_2') = \{x, y, u, v\}$, $V(D_1) - V(D_2') \neq \emptyset \neq V(D_2') - V(D_1)$ and $E(D_1) \cap E(D_2') = \{xy, xv, uy, uv\}$, let D_2' have no edge with tail y or v, and no edge with head x or u. Let D be obtained from D_0 by deleting some (possibly none) of the edges xy, xv, uy, uv, and let D_2 be obtained from D_2' by contracting the edges xy and uv. We say that D is a 4-*sum* of D_1 and D_2. We say that a digraph is *strongly planar* if it has a planar drawing such that for every vertex $v \in V(D)$, the edges of D with head v form an interval in the cyclic ordering of edges incident with v determined by the planar drawing. Let F_7 be the directed graph $D(H, M)$, where H is the Heawood graph, and M is a perfect matching of H. This defines F_7 uniquely up to isomorphism, irrespective of the choice of the bipartition of H or the choice of M. Lemma 3.3 and Theorem 4.2 imply the following.

Theorem 5.3. *A digraph D is not even if and only if it can be obtained from strongly planar digraphs and F_7 by means of* 0-, 1-, 2-, 3- *and* 4-*sums.*

From Corollary 5.1 and Lemmas 3.3 and 5.2 we deduce the following extremal result.

Corollary 5.4. *Let D be a strongly 2-connected directed graph on $n \geq 2$ vertices. If D has more than $3n - 4$ edges, then D is even.*

Corollary 5.4 does not hold for strongly connected digraphs. However, Thomassen [59] has shown that every strongly connected directed graph with minimum in- and out-degree at least three is even. This is equivalent to the following by Lemma 3.3.

Corollary 5.5. *Let G be a* 1*-extendable bipartite graph such that every vertex has degree at least four. Then G does not have Pfaffian orientation.*

If G is a brace, then the corollary follows from Corollary 5.1; otherwise the corollary follows by induction using the matching decomposition explained in the next section. The details may be found in [50, Corollary 7.8].

In [33] McCuaig used Theorem 4.2 to answer a question of Thomassen [58] by proving the following.

Theorem 5.6. *The digraph F_7 is the unique strongly* 2*-connected digraph with no even cycle.*

6. Matching decomposition

We have seen in the preceding sections that the problem of understanding which bipartite graphs are Pfaffian is reasonably well-understood and has applications outside of this subfield. We now turn our attention to the same question for general graphs. This problem seems much harder, but there are some interesting and unexpected connections.

The brick decomposition procedure of Lovász and Plummer [32] can be used to reduce the question of characterizing Pfaffian graphs to "bricks". The purpose of this section is to give an overview of this decomposition technique and to discuss recent additions to it.

A graph is *matching covered* if it is connected and every edge belongs to a perfect matching. Clearly, when deciding whether a graph G is Pfaffian we may assume that G is matching covered, for edges that belong to no perfect matching may be deleted without affecting the outcome.

Let G be a graph, and let $X \subseteq V(G)$. We use $\delta(X)$ to denote the set of edges with one end in X and the other in $V(G) - X$. A *cut* in G is any set of the form $\delta(X)$ for some $X \subseteq V(G)$. A cut C is *tight* if $|C \cap M| = 1$ for every perfect matching M in G. Every cut of the form $\delta(\{v\})$ in a graph with a perfect matching is tight; those are called *trivial*, and all other tight cuts are called *nontrivial*.

Here are three important examples of tight cuts. Let G be a matching covered graph. Assume first that G is bipartite with bipartition (A, B), and that G is not a brace. Then by Hall's theorem there is a set $X \subseteq A$ such that $|N(X)| = |X| + 1$ and $N(X) \neq B$, where $N(X)$ denotes the set of all vertices $v \in V(G) - X$ with a neighbor in X. Then $\delta(X \cup N(X))$ is a nontrivial tight cut. Now assume that G is not bipartite. If there exist distinct vertices $u, v \in V(G)$ such that $G\backslash u\backslash v$ has no perfect matching, then by Tutte's 1-factor theorem [61] there exists a nonempty set $X \subseteq V(G)$ such that $G\backslash X$ has exactly $|X|$ odd components. Furthermore, by repeatedly adding to X one vertex from each even component of $G\backslash X$ we may assume that $G\backslash X$ has no even components. Since G is matching covered no edge of G has both ends in X, and since G is not bipartite some component of $G\backslash X$, say C, has more than one vertex. But then $\delta(V(C))$ is a nontrivial tight cut. Finally, if G is not 3-connected, then let u, v be distinct vertices of G such that $G\backslash u\backslash v$ is disconnected. Let A be the vertex-set of a component of $G\backslash u\backslash v$ and let B be the union of all the remaining components. Notice that if $|A|$ is odd, then $G\backslash u\backslash v$ has no perfect matching. If $|A|$ is even, then $\delta(A \cup \{u\})$ is a nontrivial tight cut.

It is not true that every nontrivial tight cut arises as described above, but Theorem 6.1 below implies that if a graph has a nontrivial tight cut, then it has a nontrivial tight cut that arises in one of the ways described in the previous paragraph. A *brick* is a 3-connected graph G such that $G\backslash u\backslash v$ has a perfect matching for every two distinct vertices u, v of G.

Let $\delta(X)$ be a nontrivial tight cut in a graph G, let G_1 be obtained from G by identifying all vertices in X into a single vertex and deleting all resulting parallel edges, and let G_2 be defined analogously by identifying all vertices in $V(G)-X$. Then many matching-related problems can be solved for G if we are given the corresponding solutions for G_1 and G_2.

The above decomposition process can be iterated, until we arrive at graphs with no nontrivial tight cuts. Lovász [31] proved that the list of indecomposable graphs obtained at the end of the procedure does not depend on the choice of tight cuts made during the process. These indecomposable graphs were characterized by Edmonds, Lovász and Pulleyblank [12], [13]:

Theorem 6.1. *Let G be a matching covered graph. Then G has no nontrivial tight cut if and only if G is a brick or a brace.*

In light of this theorem and the previous discussion we say that a brick or a brace H is a *brick* or a *brace of a graph* G if H is obtained when the tight cut decomposition procedure is applied to G.

Vazirani and Yannakakis [64] used the tight cut decomposition procedure to reduce the study of Pfaffian graphs to bricks and braces:

Theorem 6.2. *A graph G is Pfaffian if and only if every brick and brace of G is Pfaffian.*

In particular, this justifies our earlier claim that in order to understand Pfaffian bipartite graphs it suffices to understand Pfaffian braces. Since Pfaffian braces are characterized by Theorem 4.2, in order to understand Pfaffian graphs it suffices to understand Pfaffian bricks. We return to this problem later, but in the remainder of this section we describe a characterization of bricks, developed for the purpose of studying Pfaffian bricks. We need a few definitions first.

Let G be a graph, and let v_0 be a vertex of G of degree two incident with the edges $e_1 = v_0v_1$ and $e_2 = v_0v_2$. Let H be obtained from G by contracting both e_1

and e_2 and deleting all resulting parallel edges. We say that H was obtained from G by *bicontracting* or *bicontracting the vertex* v_0, and write $H = G/v_0$. Let us say that a graph H is a *reduction* of a graph G if H can be obtained from G by deleting an edge and bicontracting all resulting vertices of degree two. By a *prism* we mean the unique 3-regular planar graph on six vertices. The following is a generation theorem of de Carvalho, Lucchesi and Murty [9].

Theorem 6.3. *If G is a brick other than K_4, the prism, and the Petersen graph, then some reduction of G is a brick other than the Petersen graph.*

Thus if a brick G is not the Petersen graph, then the reduction operation can be repeated until we reach K_4 or the prism. By reversing the process Theorem 6.3 can be viewed as a generation theorem.

Theorem 6.3 has interesting applications. First of all, it implies several results about various spaces generated by perfect matchings, including a deep theorem of Lovász [31] that characterizes the matching lattice of a graph. Second, it implies Theorem 3.1 (more precisely the most difficult part of that theorem, namely that it holds for bricks). Third, it can be used to prove a uniqueness theorem for Pfaffian orientations [8]:

Theorem 6.4. *A Pfaffian orientation of a graph G can be transformed to any other Pfaffian orientation of G by repeatedly applying the following operations:*
(1) *reversing the direction of all edges of a cut of G,*
(2) *reversing all edges with both ends in S for some tight cut $\delta(S)$,*
(3) *reversing the direction of all edges of G.*

There is a strengthening of Theorem 6.3, which we now describe. First, the starting graph can be any matching minor of G except K_4 and the prism, and second, reduction can be replaced by a more restricted operation, the following. We say that a graph H is a *proper reduction* of a graph G if it is a reduction in such a way that the bicontractions involved do not produce parallel edges. Unfortunately, Theorem 6.3 does not hold for proper reductions, but all the exceptions can be conveniently described. Let us do that now.

Let C_1 and C_2 be two vertex-disjoint cycles of length $n \geq 3$ with vertex-sets $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ (in order), respectively, and let G_1 be the graph obtained from the union of C_1 and C_2 by adding an edge joining u_i and v_i for each $i = 1, 2, \ldots, n$. We say that G_1 is a *planar ladder*. Let G_2 be the graph consisting of a cycle C with vertex-set $\{u_1, u_2, \ldots, u_{2n}\}$ (in order), where $n \geq 2$ is an integer, and n edges with ends u_i and u_{n+i} for $i = 1, 2, \ldots, n$. We say that G_2 is a *Möbius ladder*. A *ladder* is a planar ladder or a Möbius ladder. Let G_1 be a planar ladder as above on at least six vertices, and let G_3 be obtained from G_1 by deleting the edge u_1u_2 and contracting the edges u_1v_1 and u_2v_2. We say that G_3 is a *staircase*. Let $t \geq 2$ be an integer, and let P be a path with vertices $v_1, v_2, \ldots, v_t$ in order. Let G_4 be obtained from P by adding two distinct vertices x, y and edges xv_i and yv_j for $i = 1, t$ and all even $i \in \{1, 2, \ldots, t\}$ and $j = 1, t$ and all odd $j \in \{1, 2, \ldots, t\}$.

Let G_5 be obtained from G_4 by adding the edge xy. We say that G_5 is an *upper prismoid*, and if $t \geq 4$, then we say that G_4 is a *lower prismoid*. A *prismoid* is a lower prismoid or an upper prismoid. We are now ready to state a strengthening of Theorem 6.3, proved in [43].

Theorem 6.5. *Let H, G be bricks, where H is isomorphic to a matching minor of G. Assume that H is not isomorphic to K_4 or the prism, and G is not a ladder, wheel, staircase or prismoid. Then a graph isomorphic to H can be obtained from G by repeatedly taking proper reductions in such a way that all the intermediate graphs are bricks not isomorphic to the Petersen graph.*

As a counterpart to Theorem 6.5, [43] describes the starting graphs for the generation process. Notice that K_4 is a wheel, a Möbius ladder, a staircase and an upper prismoid, and that the prism is a planar ladder, a staircase and a lower prismoid.

Theorem 6.6. *Let G be a brick not isomorphic to K_4, the prism or the Petersen graph. Then G has a matching minor isomorphic to one of the following seven graphs: the graph obtained from the prism by adding an edge, the lower prismoid on eight vertices, the staircase on eight vertices, the staircase on ten vertices, the planar ladder on ten vertices, the wheel on six vertices, and the Möbius ladder on eight vertices.*

If H is a brick isomorphic to a matching minor of a brick G and G is a ladder, wheel, staircase or prismoid, then H itself is a ladder, wheel, staircase or prismoid, and can be obtained from a graph isomorphic to G by taking (improper) reductions in such a way that all intermediate graphs are bricks. Thus Theorems 6.5 and 6.6 imply Theorem 6.3. Theorems 6.5 and 6.6 were used to prove two results about minimal bricks [42], and to generate interesting examples of Pfaffian bricks. We will discuss some of those later.

McCuaig [34] proved an analogue of Theorem 6.5 for braces and used it in his proof of Theorem 4.2 in [35]. To state his result we need another exceptional class of graphs. Let C be an even cycle with vertex-set $v_1, v_2, \ldots, v_{2t}$ in order, where $t \geq 2$ is an integer and let G_6 be obtained from C by adding vertices v_{2t+1} and v_{2t+2} and edges joining v_{2t+1} to the vertices of C with odd indices and v_{2t+2} to the vertices of C with even indices. Let G_7 be obtained from G_6 by adding an edge $v_{2t+1}v_{2t+2}$. We say that G_7 is an *upper biwheel*, and if $t \geq 3$ we say that G_6 is a *lower biwheel*. A *biwheel* is a lower biwheel or an upper biwheel. McCuaig's result is as follows.

Theorem 6.7. *Let H, G be braces, where H is isomorphic to a matching minor of G. Assume that if H is a planar ladder, then it is the largest planar ladder matching minor of G, and similarly for Möbius ladders, lower biwheels and upper biwheels. Then a graph isomorphic to H can be obtained from G by repeatedly taking proper reductions in such a way that all the intermediate graphs are braces.*

Actually, Theorem 6.7 follows from a stronger version of Theorem 6.5 proved in [43].

7. Crossing numbers and k-Pfaffian graphs

By a drawing Γ of a graph in a surface Σ we mean an immersion of G in Σ such that edges are represented by homeomorphic images of [0, 1], not containing vertices in their interiors. Edges are permitted to intersect, but there are only finitely many intersections and each intersection is a crossing. For edges e, f of a drawing Γ let $cr(e, f)$ denote the number of times the edges e and f cross. For a perfect matching M let $cr_\Gamma(M)$, or $cr(M)$ when Γ is understood from the context, denote $\sum cr(e, f)$, where the sum is taken over all unordered pairs of distinct edges $e, f \in M$. The following theorem was proved by Norine [38]. The "if" part was known to Kasteleyn [23] and was proved by Tesler [56]. Norine's proof of that implication is different.

Theorem 7.1. *A graph G is Pfaffian if and only if there exists a drawing of G in the plane such that $cr(M)$ is even for every perfect matching M of G.*

The theory of crossing numbers is fairly well-developed, but only few results involve parity of crossing numbers, and I am not aware of any about crossings of perfect matchings. The closest relative of Theorem 7.1 seems to be the following classical result of Hanani [20] and Tutte [62].

Theorem 7.2. *Let Γ be a drawing of a graph in the plane such that $cr(e, f)$ is even for every two distinct non-adjacent edges e, f of G. Then G is planar.*

In fact, there is a deeper connection between the last two theorems. Norine [40] generalized Theorem 7.1 to a statement about the parity of self-intersections of different T-joins of a graph, and this generalization implies Theorem 7.2 as well as other results about crossing numbers. We omit the precise statement and instead refer the readers to [40].

A graph G is *k-Pfaffian* if there exist orientations $D_1, D_2, \ldots, D_k$ of G and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\sum_{i=1}^{k} \alpha_i \operatorname{sgn}_{D_i}(M) = 1$ for every perfect matching M of G. Thus if k is fixed and the orientations and coefficients as above are given, then the number of perfect matchings of G can be calculated efficiently, using Lemma 2.1. The following was noted by Kasteleyn [23] and proved by Galluccio and Loebl [17] and Tesler [56].

Theorem 7.3. *Every graph that has an embedding in the orientable surface of genus g is 4^g-Pfaffian.*

In light of Theorem 7.1 one might speculate that 4-Pfaffian graphs are related to graphs drawn on the torus subject to the parity condition of Theorem 7.1. That is indeed true, as shown by Norine [41].

Theorem 7.4. *Every 3-Pfaffian graph is Pfaffian. A graph G is 4-Pfaffian if and only if it can be drawn on the torus such that $cr(M)$ is even for every perfect matching M of G.*

It is therefore sensible to conjecture a generalization to surfaces of arbitrary genus, as does Norine in [41]:

Conjecture 7.5. For a graph G and integer $g \geq 0$ the following conditions are equivalent:

(1) There exists a drawing of G on the orientable surface of genus g such that $cr(M)$ is even for every perfect matching M of G.

(2) The graph G is 4^g-Pfaffian.

(3) The graph G is $(4^{g+1} - 1)$-Pfaffian.

Norine [41] has shown that every 5-Pfaffian graph is 4-Pfaffian, but his method breaks down after that.

8. Signs of edge-colorings

In this section we relate signs of edge-colorings (as in Penrose [46]) with "Pfaffian labelings", a generalization of Pfaffian orientations, whereby edges are labeled by elements of an Abelian group with an element of order two.

A graph G is called *k-list-edge-colorable* if for every set system $\{S_e : e \in E(G)\}$ such that $|S_e| = k$ there exists a proper edge coloring c with $c(e) \in S_e$ for every $e \in E(G)$. The following famous list-edge-coloring conjecture was suggested independently by various researchers and first appeared in print in [5].

Conjecture 8.1. Every k-edge-colorable graph is k-list-edge-colorable.

In a k-regular graph G one can define an equivalence relation on k-edge colorings as follows. Let $c_1, c_2 \colon E(G) \to \{1, \ldots, k\}$ be two (proper) k-edge colorings of G. For $v \in V(G)$ let $\pi_v \colon \{1, \ldots, k\} \to \{1, \ldots, k\}$ be the permutation such that $\pi_v(c_1(e)) = c_2(e)$ for every $e \in E(G)$ incident with v, and let $c_1 \sim c_2$ if $\prod_{v \in V(G)} sgn(\pi_v) = 1$. Obviously $\sim$ is an equivalence relation on the set of k-edge colorings of G and $\sim$ has at most two equivalence classes. We say that c_1 and c_2 *have the same sign* if $c_1 \sim c_2$ and we say that c_1 and c_2 *have opposite signs* otherwise.

A powerful algebraic technique developed by Alon and Tarsi [2] implies [1] that if in a k-edge-colorable k-regular graph G all k-edge colorings have the same sign then G is k-list-edge-colorable. In [14] Ellingham and Goddyn prove the following theorem.

Theorem 8.2. *In a k-regular planar graph all k-edge colorings have the same sign. Therefore every k-edge-colorable k-regular planar graph is k-list-edge-colorable.*

Goddyn [18] conjectured that Theorem 8.2 generalizes to Pfaffian graphs. This turned out to be true, even for the somewhat larger class of graphs that admit Pfaffian labelings. Let us introduce those graphs now.

Let Γ be an Abelian multiplicative group, let 1 be the identity of Γ and let -1 be some element of order two in Γ. Let G be a graph with $V(G) = \{1, 2, \ldots, n\}$, and let D be the orientation of G in which every edge ij is oriented from i to j, where $i < j$. Let us recall that $\text{sgn}_D(M)$ was defined in Section 2. We say that $l: E(G) \to \Gamma$ is a *Pfaffian labeling* of G if for every perfect matching M of G, $\text{sgn}_D(M) = \prod_{e \in M} l(e)$. We say that G admits a *Pfaffian* Γ*-labeling* if there exists a Pfaffian labeling $l: E(G) \to \Gamma$ of G. We say that G admits a *Pfaffian labeling* if G admits a Pfaffian Γ-labeling for some Abelian group Γ as above. It is easy to see that a graph G admits a Pfaffian $\mathbb{Z}_2$-labeling if and only if G admits a Pfaffian orientation. Note also that the existence of a Pfaffian labeling of a graph does not depend on the ordering of its vertices. The results of the remainder of this section are from [44].

Theorem 8.3. *Let G be a k-regular graph with $V(G) = \{1, \dots, 2n\}$. If G admits a Pfaffian labeling then all k-edge-colorings of G have the same sign.*

Using the theory of Alon and Tarsi mentioned above this implies a proof of Goddyn's conjecture:

Corollary 8.4. *Every k-edge-colorable k-regular graph that admits a Pfaffian labeling is k-list-edge-colorable.*

Theorem 8.3 has the following partial converse.

Theorem 8.5. *If a graph G does not admit a Pfaffian labeling then there exist an integer k, a k-regular multigraph G' whose underlying simple graph is a spanning subgraph of G and two k-edge colorings of G' of different signs.*

The above two theorems suggest the study of graphs that admit a Pfaffian labeling. First, there is an analogue of Theorem 6.2.

Lemma 8.6. *Let Γ be an Abelian group. A matching covered graph G admits a Pfaffian Γ-labeling if and only if each of its bricks and braces admits a Pfaffian Γ-labeling.*

Thus it suffices to characterize bricks and braces that have a Pfaffian labeling. The Petersen graph is not Pfaffian, but it admits a Pfaffian μ_4-labeling, where μ_n is the multiplicative group of the n^{th} roots of unity. Figure 2 shows an example of such a labeling. Using Theorems 6.3 and 6.7 it is not hard to show that the Petersen graph is the only brick or brace with that property. Using Lemma 8.6 we obtain:

Theorem 8.7. *A graph G admits a Pfaffian labeling if and only if every brick and brace in its decomposition is either Pfaffian or isomorphic to the Petersen graph. If G admits a Pfaffian Γ-labeling for some Abelian group Γ then G admits a Pfaffian μ_4-labeling.*

The last result of this section characterizes graphs that admit a Pfaffian labeling in terms of their drawing in the projective plane. We say that a region C of the projective plane is *a crosscap* if its boundary is a simple closed curve and its complement is a disc. We say that a drawing Γ of a graph G in the projective plane is *proper with*

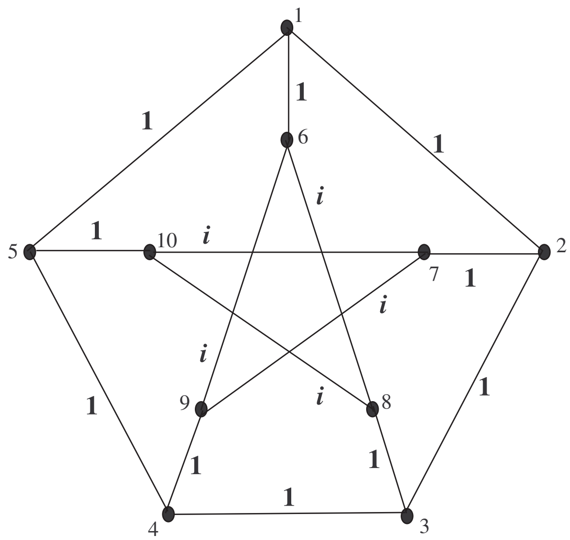

Figure 2. A μ_4-labeling of the Petersen graph.

respect to the crosscap C if no vertex of G is mapped to C and for every edge $e \in E(G)$ intersecting C and every crosscap $C' \subseteq C$ the image of e intersects C'.

Theorem 8.8. *A graph G admits a Pfaffian labeling if and only if there exists a crosscap C in the projective plane and a proper drawing Γ of G in the projective plane with respect to C such that $|M \cap S|$ and $cr_\Gamma(M)$ are even for every perfect matching M of G, where $S \subseteq E(G)$ denotes the set of edges whose images intersect C.*

9. On characterizing Pfaffian graphs

Norine's theorem, Theorem 7.1, is a beautiful result, but, unfortunately, does not seem to help testing whether a graph is Pfaffian. Theorems 4.1 and 4.2 suggest two possible ways of characterizing Pfaffian graphs, but neither has been carried out, and there appear to be serious difficulties.

Fischer and Little [15] extended Theorem 4.1 as follows. A matching covered graph is *near bipartite* if it has two edges whose deletion makes the graph bipartite and matching covered. Let *Cubeplex* be the graph obtained from the (skeleton of the 3-dimensional) cube by subdividing three edges of a perfect matching and adding a vertex of degree three adjacent to the three resulting vertices, and let *Twinplex* be obtained from the Petersen graph by subdividing two edges that form an induced matching and joining the resulting vertices by an edge. This defines both graphs uniquely up to isomorphism. We say that a graph is a *weak matching minor* of another if the first can be obtained from a matching minor of the second by contracting odd cycles and deleting all resulting loops and parallel edges.

Theorem 9.1. *A near bipartite graph is Pfaffian if an only if it has no weak matching minor isomorphic to $K_{3,3}$, Cubeplex or Twinplex.*

Let us say that a graph G is *minimally non-Pfaffian* if G is not Pfaffian but every proper weak matching minor of G is. Thus $K_{3,3}$, Cubeplex and Twinplex are minimally non-Pfaffian, and so is the Petersen graph, as is easily seen. Fischer and Little (private communication) conjectured that those are the only minimally non-Pfaffian graphs; in other words they conjectured that upon adding the Petersen graph to the list of excluded weak matching minors, Theorem 9.1 holds for all graphs. Unfortunately, that is not the case. Here is an infinite family of minimally non-Pfaffian graphs [45].

Let $k \geq 2$, let C_{2k+1} be the cycle of length $2k+1$ with vertices $1, 2, \ldots, 2k+1$ in order, and let M be a matching in C, possibly empty. Let the graph $G(k, M)$ be defined by saying that its vertex-set is $\{u_1, u_2, \ldots, u_{2k+1}, v_1, v_2, \ldots, v_{2k+1}, w_1, w_2\}$ and that $G(k, M)$ has the following edges, where index arithmetic is taken modulo $2k+1$:

- $u_i v_i$ for all $i = 1, 2, \ldots, 2k+1$,
- $u_i u_{i+1}$ and $v_i w_2$ if $\{i, i+1\} \notin M$,
- $v_i w_1$ if $\{i-1, i\} \notin M$,
- $u_i v_{i+1}$ and $v_i u_{i+1}$ if $\{i, i+1\} \in M$.

Notice that the graph $G(2, \{\{1, 2\}, \{3, 4\}\})$ is isomorphic to Cubeplex.

Theorem 9.2. *For every integer $k \geq 2$ and every matching M of C_{2k+1} the graph $G(k, M)$ is minimally non-Pfaffian.*

Thus an extension of Theorem 9.1 to all graphs would have to involve infinitely many excluded weak matching minors. On the other hand, as noted in [45], the family $G(k, M)$ suggests a possible weakening of the weak matching minor ordering, and it is possible that if weak matching minor is replaced by this weakening, then the list of excluded might be finite (and of a reasonable size).

There is also the possibility of extending Theorem 4.2 to all graphs. That could be potentially very profitable, because it might lead to a polynomial-time recognition algorithm, but the prospects for that are not very bright. The class of planar graphs can be enlarged to a bigger class of Pfaffian graphs defined by means of surface embeddings. Let us say that an embedding of a graph G in the Klein bottle is *cross-cap-odd* if every cycle C in G that does not separate the surface is odd if and only if it is 1-sided. If G is embedded in the Klein bottle with all faces even (that is, bounded by a walk of even length), then the embedding is cross-cap-odd if and only if the 1-sided cycles are precisely the odd cycles in G. Please note that every planar graph can be embedded in the Klein bottle so that the embedding will be cross-cap-odd, and every embedding of a non-bipartite graph in the projective plane with all faces even may be regarded as a cross-cap-odd embedding in the Klein bottle. The following result, proved in [39], resulted from earlier conversations of the author with Neil Robertson and Paul Seymour. By the remark above it implies Theorem 1.1.

Theorem 9.3. *Every graph that admits a cross-cap-odd embedding in the Klein bottle is Pfaffian.*

It may seem reasonable to expect an analogue of Theorem 4.2, something along the lines that every Pfaffian brick can be obtained from graphs that admit a cross-cap-odd embedding in the Klein bottle and a few sporadic exceptional graphs by means of certain composition operations. Unfortunately, the following construction of Norine seems to give a counterexample.

Theorem 9.4. *For every integer $n \geq 1$ there exists a Pfaffian brick that has a subgraph isomorphic to K_n.*

There is a chance that a notion analogous to tree-width can help us get around Norine's construction. A tree is *ternary* if all its vertices have degree one or three; the vertices of degree one are called *leaves*. A *matching decomposition* of a graph G is a pair (T, τ), where T is a ternary tree and τ is a bijection from the set of leaves of T to $V(G)$. For an edge $e \in E(T)$ fix one of the two components of $T \backslash e$, and let V_e be the set of all leaves of T that belong to that component. We define the *width* of e as the maximum, over all perfect matchings M of G, of $|\delta(\tau(V_e)) \cap M|$. We define the *width* of (T, τ) as the maximum width of an edge of T, and we define the *matching-width* of a graph G as the minimum width of a matching decomposition of G. The graphs Norine constructed in the proof of Theorem 9.4 all have low matching-width, and so that leaves open the possibility that Pfaffian graphs of high matching-width might exhibit more structure. Further, Norine [39] describes a polynomial-time algorithm to test whether an input graph G is Pfaffian, assuming a matching decomposition of G of bounded width is given as part of the input instance. Thus there is some hope, but at the moment it is not clear if these ideas can be turned into a meaningful theorem or a polynomial-time algorithm to test whether an input graph is Pfaffian. The following conjecture of Norine and the author [39], modeled after the excluded grid theorem of Robertson and Seymour [48] (see also [11], [49]), seems relevant.

Conjecture 9.5. There exists a function f such that every graph of matching-width at least $f(k)$ has a matching minor isomorphic to the $2k \times 2k$ grid.

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