Three-coloring triangle-free graphs on surfaces VI. 3-colorability of quadrangulations*

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Abstract

We give a linear-time algorithm to decide 3-colorability (and find a 3-coloring, if it exists) of quadrangulations of a fixed surface. The algorithm also allows to prescribe the coloring for a bounded number of vertices.

1 Introduction

This paper is a part of a series aimed at studying the 3-colorability of graphs (possibly with parallel edges, but no loops) on a fixed surface that are either triangle-free, or have their triangles restricted in some way (throughout the paper, all colorings are proper, i.e., adjacent vertices have different colors). Embeddability in a surface is not a sufficient restriction by itself, as 3-colorability of planar graphs is NP-complete [6]. Restricting the triangles is natural in the light of the well-known theorem of Grötzsch stating that every planar triangle-free graph is 3-colorable.

Quadrangulations of a surface represent an important special case of the problem, as evidenced by the following theorem of Gimbel and Thomassen [7].

Theorem 1.1. *A triangle-free graph embedded in the projective plane is* 3*-colorable if and only if it has no subgraph isomorphic to a non-bipartite quadrangulation of the projective plane.*

In a previous paper of this series [3], we showed how Theorem 1.1 can be generalized to other surfaces. A graph G is 4-*critical* if its chromatic number is 4 and the chromatic number of every proper subgraph of G is at most 3.

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Theorem 1.2 ([3], Theorem 1.3)**.** *There exists a constant κ with the following property. Let G be a graph embedded in a surface of Euler genus g. Let t be the number of triangles in G and let c be the number of 4-cycles in G that do not bound a 2-cell face. If G is 4-critical, then*

$$\sum_{f \text{ face of } G} (|f| - 4) \le \kappa(g + t + c - 1).$$

This result forms a basis for an algorithm to decide the 3-colorability of a triangle-free graph G embedded in a surface of Euler genus g. Let us for simplicity assume that all 4-cycles in G bound 2-cell faces. By Theorem 1.2, it suffices to enumerate all subgraphs H of G such that

$$\sum_{h \text{ face of } H} (|h| - 4) \le \kappa g \tag{1}$$

and test whether they are 3-colorable. There are at most $|V(G)|^{5\kappa g}$ subgraphs of G satisfying (1)—enumerate the boundaries of the faces of H of length greater than 4, and test whether all the faces of G contained in the rest have length 4. Hence, if we can test the 3-colorability of such a graph H in a polynomial time, we obtain a polynomial-time algorithm to test 3-colorability of G (let us remark that the exponent of the polynomial bounding the complexity of this algorithm depends on g; in [5], we give a more involved linear-time algorithm).

A graph H satisfying (1) contains at most $5\kappa g$ vertices incident with faces of length greater than 4. Therefore, we can try all possible 3-colorings of these vertices and test whether they extend to a 3-coloring of H. In this paper, we develop a linear-time algorithm to perform this last step.

Let us now state the result more precisely, starting with a number of definitions. A *surface* is a compact connected 2-manifold with (possibly null) boundary. Each component of the boundary is homeomorphic to a circle, and we call it a *cuff*. For non-negative integers a, b and c, let $\Sigma(a, b, c)$ denote the surface obtained from the sphere by adding a handles, b crosscaps and removing the interiors of c pairwise disjoint closed discs. The classification theorem of surfaces shows that every surface is homeomorphic to $\Sigma(a, b, c)$ for some choice of a, b and c. The *Euler genus* $g(\Sigma)$ of a surface Σ homeomorphic to $\Sigma(a, b, c)$ is defined as $2a + b$.

Consider a graph G embedded in the surface Σ; when useful, we identify G with the topological space consisting of the points corresponding to the vertices of G and the simple curves corresponding to the edges of G. A *face* f of G is a maximal connected subset of $\Sigma - G$. By the *length* $|f|$ of f, we mean the sum of the lengths of the boundary walks of f (in particular, if an edge appears twice in the boundary walks, it contributes 2 to $|f|$). A face f is *closed* 2*-cell* if it is homeomorphic to an open disk and its boundary forms a cycle in G. A graph H is a *quadrangulation* of a surface Σ if all faces of H are closed 2-cell and have length 4 (in particular, the boundary of Σ is formed by a set of pairwise vertex-disjoint cycles in H, called the *boundary cycles* of H). A vertex of G contained in the boundary of Σ is called a *boundary vertex*.

Finally, we are ready to formulate our main algorithmic result.

Theorem 1.3. *For every surface Σ and integer k, there exists a linear-time algorithm with input*

- *G: a quadrangulation of Σ with at most k boundary vertices, and*
- *ψ: a function from boundary vertices of G to $\{1,2,3\}$,*

which decides whether there exists a 3-coloring φ of G such that $\varphi(v) = \psi(v)$ for every boundary vertex v of G. In the affirmative case, the algorithm also outputs such a coloring φ.

The first step towards this algorithm was made by Hutchinson [8], who proved that for every orientable surface Σ, there exists c such that every graph embedded in Σ with all faces of even length and without non-contractible cycles of length at most c is 3-colorable. One could hope for a similar result in the setting of Theorem 1.3; however, there turns out to be one more obstruction to the existence of a 3-coloring, based on *winding number*.

Let G be graph and let ψ be its coloring by colors $\{1,2,3\}$ and let $Q = v_1v_2\ldots v_kv_1$ be a directed closed walk in G. One can view ψ as mapping Q to a closed walk in a triangle T, and the winding number of ψ on Q is then the number of times this walk goes around T in a fixed direction. More precisely, for $uv \in E(G)$, let $\delta_\psi(u,v) = 1$ if $\psi(v) - \psi(u) \in \{1,-2\}$, and $\delta_\psi(u,v) = -1$ otherwise. For a walk $W = u_1u_2\ldots u_m$, let $\delta_\psi(W) = \sum_{i=1}^{m-1} \delta_\psi(u_i,u_{i+1})$. The winding number $\omega_\psi(Q)$ of ψ on Q is defined as $\delta_\psi(Q)/3$.

Suppose that G is embeded in an orientable surface Σ so that every face of G is closed 2-cell. Let $\mathcal{C}$ be the set consisting of all facial and boundary cycles of G. For each of the cycles in $\mathcal{C}$, choose an orientation of its edges in the clockwise direction around the corresponding face or cuff, so that every edge $e \in G$ is oriented in opposite directions in the two cycles of $\mathcal{C}$ containing e. For any 3-coloring ψ of G, consider the sum of its winding numbers on the cycles in $\mathcal{C}$ in this orientation. Since each edge appears in opposite orientations in two cycles of $\mathcal{C}$, their contributions cancel out.

Observation 1.4. *Let G be a graph embedded in an orientable surface Σ so that every face of G is closed 2-cell. Let $Q_1, \ldots, Q_m$ be the cycles of $\mathcal{C}$ viewed as closed walks in the clockwise orientation. If ψ is a 3-coloring of G, then*

$$\sum_{i=1}^{m} \omega_\psi(Q_i) = 0.$$

As the winding number of any 3-coloring on a 4-cycle is 0, we obtain the following constraint on 3-colorings of quadrangulations.

Corollary 1.5. *Let G be a quadrangulation of an orientable surface Σ. Let $B_1, \ldots, B_k$ be the boundary cycles of G in the clockwise orientation. If ψ is a 3-coloring of G, then*

$$\sum_{i=1}^{k} \omega_\psi(B_i) = 0.$$

Therefore, in addition to forbidding short non-contractible cycles, for orientable surfaces we also need to require that the winding numbers of boundary cycles in their prescribed coloring sum to 0.

For non-orientable surfaces, the situation is a bit more complicated. Suppose that G is a quadrangulation of a surface, and let us fix directed closed walks $B_1, \ldots, B_k$ tracing the cuffs of G. For each facial cycle, choose an orientation arbitrarily. Let D denote the directed graph with vertex set $V(G)$ and uv being an edge of D if and only if uv is an edge of G oriented towards v in both cycles of $\mathcal{C}$ that contain it. Let $p(G, B_1, \ldots, B_k) = 2|E(D)| \bmod 4$. Note that $p(G, B_1, \ldots, B_k)$ is independent on the choice of the orientations of the 4-faces, since reversing an orientation of a 4-face with d edges belonging to D changes $2|E(D)|$ by $2(4 - 2d) \equiv 0 \pmod 4$.

Consider the sum of winding numbers of a 3-coloring ψ of G on cycles in $\mathcal{C}$. As before, the contributions of all other edges of G that do not belong to D cancel out, and since G is a quadrangulation, the winding number on any non-boundary cycle in $\mathcal{C}$ is 0. Hence,

$$\sum_{i=1}^{k} \delta_\psi(B_i) = 2 \sum_{uv \in E(D)} \delta_\psi(u, v).$$

Since $\delta_\psi(u, v) = \pm 1$ for every $uv \in E(D)$,

$$2 \sum_{uv \in E(D)} \delta_\psi(u, v) \equiv 2|E(D)| \equiv -2|E(D)| \equiv -p(G, B_1, \ldots, B_k) \pmod 4,$$

regardless of the 3-coloring ψ. Furthermore,

$$\sum_{i=1}^{k} \delta_\psi(B_i) = 3 \sum_{i=1}^{k} \omega_\psi(B_i) \equiv -\sum_{i=1}^{k} \omega_\psi(B_i) \pmod 4.$$

Therefore, we get the following necessary condition for the existence of a 3-coloring.

Observation 1.6. *Let G be a quadrangulation of a surface Σ. Let $B_1, \ldots, B_k$ be the boundary cycles of G. If ψ is a 3-coloring of G, then*

$$\sum_{i=1}^{k} \omega_\psi(B_i) \equiv p(G, B_1, \ldots, B_k) \pmod 4.$$

If a 3-coloring ψ of the boundary cycles satisfies the condition of Observation 1.6, we say that ψ is *parity-compliant*. An analogue of the result of Hutchinson [8] was obtained by Mohar and Seymour [9] and Nakamoto, Negami and Ota [10]: For every non-orientable surface Σ, there exists c such that a quadrangulation G of Σ without non-contractible cycles of length at most c is 3-colorable if and only if $p(G) = 0$.

The core of our algorithm is based on the fact that the winding-number conditions are not only necessary, but also sufficient for a precoloring of boundary cycles to extend, unless the graph contains a small subgraph H such that cutting along H simplifies the surface. Let G be 2-cell embedded in a surface Σ. A subgraph H of G is *non-essential* if there exists $\Lambda \subset \Sigma$ containing H, where Λ is either an open disk, or an open disk with a hole whose boundary is equal to a cuff C of Σ. A subgraph H of G is *essential* if it is not non-essential. A cycle K is *contractible* if there exists a closed disk $\Delta \subseteq \Sigma$ with boundary equal to K. For a cuff C, let $\Sigma + \hat{C}$ denote the surface obtained from Σ by adding an open disk disjoint from Σ and with boundary equal to C. A cycle K *surrounds a cuff* C if K is not contractible in Σ, but it is contractible in $\Sigma + \hat{C}$. An embedding of G is *boundary-linked* if

- Σ is a disk and G contains no path P with both ends u and v in the boundary cycle B, such that both paths between u and v in B are strictly longer than P; or,
- Σ is a cylinder with boundary cycles B_1 and B_2 and the length of every non-contractible cycle in G distinct from B_1 and B_2 is *strictly greater* than $\max(|B_1|, |B_2|)$; or,
- Σ is neither a disk nor a cylinder and for every boundary cycle B of G, every cycle K of G surrounding the cuff incident with B has length *at least* $|B|$.

We say that a coloring ψ of the boundary cycles of a quadrangulation of a surface Σ *satisfies the winding number constraint* if either

- Σ is orientable and the sum of winding numbers of ψ on the boundary cycles of G in their clockwise orientation is 0, or
- Σ is non-orientable and ψ is parity-compliant.

Theorem 1.7. *For every surface Σ and every integer k, there exists an integer ν with the following property. Let G be a boundary-linked quadrangulation of a surface Σ, such that every connected essential subgraph of G has at least ν edges and every boundary cycle has length at most k. Let ψ be a 3-coloring of the boundary cycles of G. Then ψ extends to a 3-coloring of G if and only if it satisfies the winding number constraint.*

In order to apply Theorem 1.7, the following characterization of connected essential subgraphs by Robertson and Seymour [12] is useful. A *dumbbell* is a graph consisting either of two cycles with exactly one vertex in common, or two disjoint cycles and a path joining the two cycles and disjoint from them except for its ends. A *theta graph* is a graph consisting of three internally disjoint paths joining the same pair of distinct vertices. A *lollipop* is a graph consisting of a cycle and a path (the path may be just a single vertex) with one end on the cycle and otherwise disjoint from the cycle. The other end of the path will be called the *tip* of the lollipop. A path P is a *spoke* if it intersects the boundary exactly in its endpoints and both of them belong to the same boundary cycle B. A *base* of the spoke is a subpath Q of B with the same endpoints as P such that Q is homotopic to P. In a disk, every spoke has two bases, while in any other surface, it has at most one base.

Lemma 1.8. *Let G be a quadrangulation of a surface Σ and let H be an inclusion-wise minimal connected essential subgraph of G. Then H satisfies one of the following conditions:*

(i) H is a path joining boundary vertices of distinct cuffs and containing no other boundary vertices, or

(ii) H is a cycle containing at most one boundary vertex and H is neither contractible nor surrounds a cuff, or

(iii) H is a spoke with no base, or

(iv) H is a dumbbell or a theta graph containing no boundary vertices and the cycles of H surround pairwise distinct cuffs of Σ, or

(v) H is a lollipop with the tip in a cuff C, with no other boundary vertex, and the cycle of H surrounds a cuff distinct from C.

If Σ is a cylinder, then only (i) is possible.

Observe that cutting along a connected essential subgraph simplifies the surface (decreases its genus or number of cuffs). In combination with the results of [3], Theorem 1.7 implies an important structural observation. For a quadrangulation G of a surface Σ, a subgraph H of G and a face h of H, let Σ_h denote a surface whose interior is homeomorphic to h, let $\theta_h : \Sigma_h \to \Sigma$ be a continuous function whose restriction to the interior of Σ_h is a homeomorphism to h, and let $G_h = \theta_h^{-1}(G)$.

Theorem 1.9. *For every surface Σ and an integer $k \geq 0$, there exists a constant β with the following property. Let G be a triangle-free graph embedded in Σ so that every cuff of Σ traces a cycle in G and so that the sum of the lengths of the boundary cycles of G is at most k. Suppose that every contractible 4-cycle in G bounds a face. Then G has a subgraph H with at most β vertices, such that H contains all the boundary cycles and each face h of H satisfies one of the following.*

(a) Every precoloring of the boundary of h extends to a 3-coloring of G_h, or

(b) G_h is a quadrangulation and every precoloring of the boundary of H which satisfies the winding number constraint extends to a 3-coloring of G_h, or

(c) h is an open cylinder and G_h is its quadrangulation, or

(d) h is an open cylinder and both boundary cycles of h have length exactly 4.

Let us remark that if Σ is the disk, the outcome (d) of the theorem cannot occur, since otherwise G would contain a non-facial contractible 4-cycle. As a corollary of Theorem 1.9, we obtain a strengthening of the results of Hutchinson [8], Mohar and Seymour [9] and Nakamoto, Negami and Ota [10].

Corollary 1.10. *For every surface Σ without boundary, there exists a constant γ with the following property. Let G be a triangle-free graph embedded in Σ. If Σ is non-orientable, assume furthermore that no subgraph H of G is a quadrangulation with $p(H) \neq 0$. If the edge-width of the embedding of G is at least γ, then G is 3-colorable.*

In the previous paper of the series [4] (Lemma 4.5), we proved Theorem 1.7 in the special case that Σ is a cylinder. In Section 2, we prove a generalization of Theorem 1.7 in the special case that Σ is a disk. In Section 3, we prove the general case of Theorem 1.7, and in Section 4, we use it to derive the algorithm of Theorem 1.3. Finally, in Section 5, we prove Theorem 1.9 and show that it implies Corollary 1.10.

2 The disk

While in general Theorem 1.7 only gives a sufficient condition for the existence of a 3-coloring, in the special case of a quadrangulation of the disk, we can give an exact characterization.

Lemma 2.1. *Let G be a quadrangulation of a disk Σ with the boundary cycle B. Let ψ be a 3-coloring of B with winding number 0. Then ψ extends to a 3-coloring of G if and only if*

($\star$) *every spoke P of Σ and each base Q of P satisfy $|P| \geq |\delta_\psi(Q)|$.*

Proof. Let us first consider the case that B contains a spoke P and its base Q such that $|P| < |\delta_\psi(Q)|$. Let H be the subgraph of G drawn in the closed disk Δ bounded by the cycle $C = Q \cup P$. Suppose that ψ extends to a 3-coloring φ of G, and consider the restriction of φ to H. We have $\omega_\varphi(C) = \frac{1}{3}\delta_\varphi(C) = \frac{1}{3}(\delta_\psi(Q) + \delta_\varphi(P))$. However, $|\delta_\psi(Q) + \delta_\varphi(P)| \geq |\delta_\psi(Q)| - |\delta_\varphi(P)| \geq |\delta_\psi(Q)| - |P| > 0$, and thus φ does not have zero winding number on C. This contradicts Corollary 1.5, and thus no 3-coloring extends ψ.

Next, let us consider the case that ($\star$) holds. Let $B = b_1 b_2 \dots b_k$, let S be the set of edges $b_i b_{i+1} \in E(B)$ such that $\psi(b_{i+1}) - \psi(b_i) \in \{1, -2\}$, where $b_{k+1} = b_1$, and let $T = E(B) \setminus S$. Note that since ψ has zero winding number on B, we have $|S| = |T|$. Let Π be the sphere obtained from the disk containing G by adding a disjoint open disk Λ with the boundary B. Let $G^\star$ be the dual of G in Π. Let $S^\star$ and $T^\star$ denote the sets of edges of $G^\star$ corresponding to the edges of S and T, respectively. let H be the graph obtained from $G^\star$ by splitting the vertex corresponding to Λ to two non-adjacent vertices s and t, where s is incident with the edges of $S^\star$ and t is incident with the edges of $T^\star$.

Suppose for a contradiction that H contains an edge-cut K separating s from t such that $|K| < |S^\star|$. Let K_0 consist of edges of K not incident with $\{s, t\}$ and let us choose K so that $|K_0|$ is as small as possible. Let Z_s and Z_t be connected components of $H - \{s, t\} - K$ incident with an edge of $S^\star \setminus K$ and an edge of $T^\star \setminus K$, respectively. Let $K_1 \subseteq K_0$ consist of edges incident with a vertex of Z_s, and let $K_2 \subset K$ consist of edges incident with a vertex of Z_s and with t. Since $H - \{s,t\}$ is connected, it follows that $K_1 \neq \emptyset$. Let $S_1 \subseteq S^\star$ consist of edges incident with a vertex of Z_s and with s. Note that $K' = (K \setminus (K_1 \cup K_2)) \cup S_1$ is an edge-cut separating s from t, and since K_1 is nonempty and K contains as few edges not incident with s or t as possible, K' must have size at least $|S^\star|$; in particular, $|K'| > |K|$, and thus $|S_1| > |K_1|+|K_2|$. Furthermore, note that $K'' = K_1 \cup K_2 \cup (S^\star \setminus S_1)$ is an edge-cut separating s from t and $|K''| = |S^\star|-|S_1|+|K_1|+|K_2| < |S^\star|$. Since K was chosen with $|K_0|$ minimal, it follows that $K_0 = K_1$; hence, all edges of K_0 are incident with Z_s. Symmetrically, all edges of K_0 are incident with Z_t, and thus $H - \{s,t\} - K$ has exactly two components Z_s and Z_t. Let P be the subgraph of G with edges corresponding to those in K_0. Since $H - \{s,t\} - K$ has exactly two components, the graph $B + P$ (drawn in the disk Σ) has exactly two faces, i.e., P is a spoke of B. Let Q be the subpath of B such that $E(Q) = K_2 \cup S_1$. Note that $||S_1| - |K_2|| = |\delta_\psi(Q)|$. However, $|P| = |K_0| = |K \setminus (S^\star \setminus S_1) \setminus K_2| = |K| - |S^\star| + |S_1| - |K_2| < |S_1| - |K_2| \leq |\delta_\psi(Q)|$, which contradicts $(\star)$.

We conclude that every edge-cut in H separating s from t has size at least $|S^\star|$. By Menger's theorem, H contains pairwise edge-disjoint paths P_1, ..., $P_{|S^\star|}$ joining s with t. Note that all vertices of $H' = H - E(P_1 \cup P_2 \cup \ldots \cup P_{|S^\star|})$ have even degree, and thus H' is a union of pairwise edge-disjoint cycles C_1, ..., C_m. For $1 \leq i \leq m$, direct the edges of C_i so that all vertices of C_i have outdegree 1. For $1 \leq i \leq |S^\star|$, direct the edges of P_i so that all its vertices except for t have outdegree 1. This gives an orientation of H such that the indegree of every vertex of $V(H) \setminus \{s,t\}$ equals its outdegree, s has indegree 0 and t has outdegree 0. Consider the corresponding orientation of $G^\star$; since $|S^\star| = |T^\star|$, the indegree of the vertex corresponding to Λ equals its outdegree. Therefore, the orientation defines a nowhere-zero Z_3-flow in $G^\star$. By Tutte [13], this nowhere-zero Z_3-flow corresponds to a 3-coloring φ of G, and the orientations of the edges of $S^\star$ and $T^\star$ were chosen so that $\varphi \restriction B$ is equal to ψ (up to permutation of colors). □

Note that if G is boundary-linked, then for every spoke P and its base Q, we have $|P| \geq |Q| \geq |\delta_\psi(Q)|$, and thus Lemma 2.1 implies Theorem 1.7 when Σ is a disk.

Corollary 2.2. *Let G be a boundary-linked quadrangulation of a disk. Let ψ be a 3-coloring of the boundary cycle B of G. Then ψ extends to a 3-coloring of G if and only if it satisfies the winding number constraint (that is, the winding number of ψ on B is 0).*

3 General surfaces

We will need the following simple observation.

Lemma 3.1. *Let $R = r_0r_1\ldots r_n$ be a path of even length, let w be an even integer such that $|w| \leq |R|$, and let $c_0, c_n \in \{1,2,3\}$ satisfy $c_n \equiv c_0 + w \pmod 3$. Then there exists a 3-coloring $\varphi : V(R) \to \{1,2,3\}$ such that $\varphi(r_0) = c_0$, $\varphi(r_n) = c_n$ and $\delta_\varphi(R) = w$.*

Proof. We prove the claim by induction on n. If $n > |w|$, then by the induction hypothesis, there exists a 3-coloring φ of $R' = R - \{r_0, r_1\}$ such that $\varphi(r_2) = c_0$, $\varphi(r_n) = c_n$ and $\delta_\varphi(R') = w$. To obtain a requested 3-coloring of R, we set $\varphi(r_0) = c_0$ and choose $\varphi(r_1)$ arbitrarily.

Hence, we can assume that $n = |w|$. Then, the unique 3-coloring φ such that $\varphi(r_0) = c_0$ and $\delta_\varphi(r_i, r_{i+1}) = \mathrm{sgn}(w)$ for $i = 0, \ldots, n-1$ satisfies $\varphi(r_n) = c_n$ and $\delta_\varphi(R) = w$. $\square$

First, we prove a weak variant of Theorem 1.7. A 3-coloring of a cycle C is *(d, k)-tame* if

- each two vertices at distance exactly three in C have different colors, and
- there exist paths $P_1, P_2, \ldots, P_t \subset C$ of length at least d whose vertex sets form a partition of $V(C)$, such that $t \leq k$ and only two colors are used on each of the paths.

Lemma 3.2. *For every surface Σ and every integer k, there exist integers d and ρ with the following property. Let G be a boundary-linked quadrangulation of Σ, such that every cycle that surrounds a cuff as well as every connected essential subgraph in G has at least ρ edges. Let ψ be a 3-coloring of the boundary cycles of Σ that is (d, k)-tame on each of the cycles. If ψ satisfies the winding number constraint, then it extends to a 3-coloring of G.*

Proof. Let g be the Euler genus of Σ and c the number of cuffs of Σ. By Grötzsch's Theorem, Theorem 1.1 and Corollary 2.2, the claim holds (with $d = \rho = 0$) if Σ is the sphere, the projective plane or the disk. Hence, assume that $g + c \geq 2$. Let $b = g + c - 1$, $\mu = 12b + 2kc$, $\beta = 12b\mu$, $\lambda = \mu + 2\beta$, $\rho = 2b\lambda$ and $d = \beta + 4\mu$.

Let H be a subgraph of G with as few edges as possible such that H contains all boundary cycles of G and every face of H is homeomorphic to an open disk. Observe that H is connected and has exactly one face Λ. By cutting along H, we obtain an embedding of a graph G' in a closed disk Δ together with a continuous surjection $\theta : \Delta \to \Sigma$ mapping G' to G such that the restriction of θ to the interior of Δ is a homeomorphism to Λ. Furthermore, θ maps the cycle Γ of G' forming the boundary of Δ to the boundary walk of the face Λ of H.

Since $g + c \geq 2$, H has minimum degree at least two and H is not a cycle. Let X be the set of vertices of H of degree at least three and let $\mathcal{P}$ be the set of all subgraphs P of H such that either P is a path in H joining two vertices $u, v \in X$, or P is a cycle containing a vertex $u = v \in X$, and such that no other vertex of P belongs to X and P is not a part of a boundary cycle. If P has length less than λ, then let $M(P)$ be the null graph. Otherwise, let $M(P)$ be a subpath of P of length μ chosen so that the distance between the endvertices of $M(P)$ and $\{u, v\}$ in P is at least β. Let M be the union of $M(P)$ over all $P \in \mathcal{P}$.

Note that by Euler's formula, H has at most $2b$ vertices of degree at least three and that $|\mathcal{P}| \leq 3b$. Let L be obtained from H by removing edges of M and of the boundary cycles, and then removing isolated vertices. Consider a path Q in H intersecting the boundary of Σ exactly in its ends. Note that if both ends of Q belong to the same boundary cycle C, then $C + Q$ does not contain a contractible cycle, since H has only one face. It follows that Q is an essential subgraph of G, and thus Q has length at least $\rho > (|X| - 1)(\lambda - 1)$. We conclude that M intersects Q, and thus L contains no path intersecting the boundary of Σ only in its ends. Similarly, since H has only one face, every cycle in H is non-contractible, and thus it has length at least ρ and intersects either the boundary of Σ or M. Therefore, L is a union of trees, each of them intersecting the boundary of Σ in at most one vertex.

If Σ is orientable, then let $M' = M$. If Σ is non-orientable, then there exists a path R forming a component of M such that for any edge uv of R, the two vertices of $\theta^{-1}(u)$ separate the two vertices of $\theta^{-1}(v)$ in the cycle Γ. In this case, we set $M' = M - R$.

We now define a 3-coloring φ of H; this will also give a 3-coloring φ' of Γ such that $\varphi'(x) = \varphi(\theta(x))$ for each $x \in V(\Gamma)$. For each vertex v incident with the boundary of Σ, we let $\varphi(v) = \psi(v)$. For each boundary cycle, fix its partition to paths as in the definition of (d, k)-tame coloring, and let $\mathcal{Q}$ be the set of these paths in all boundary cycles. We extend φ to L arbitrarily so that each component of L and the path of $\mathcal{Q}$ that intersects it (if any) is colored by exactly two colors. Next, for each path $v_0v_1 \ldots v_\mu$ of M', we extend the coloring of the component of L containing v_0 to $v_0v_1 \ldots v_{\mu-2}$ using the same two colors, and we choose $\varphi(v_{\mu-1})$ distinct from $\varphi(v_{\mu-2})$ and $\varphi(v_\mu)$.

Finally, in the case that Σ is non-orientable, we need to determine the coloring of R. Let P_1 and P_2 be the two walks obtained from the boundary walk of Γ by removing the edges and the internal vertices of $\theta^{-1}(R)$. Note that $|P_1| + |P_2| = |\Gamma| - 2|R|$ is even. For $i \in \{1, 2\}$ let u_i and v_i be the first and the last vertex of P_i, respectively, let $w_i = \delta_{\varphi'}(P_i)$ and let $w = w_1 + w_2$. Since ψ is parity-compliant, w is divisible by 4. By the choice of R, we have $\theta(u_1) = \theta(u_2)$ and $\theta(v_1) = \theta(v_2)$, and thus $\varphi'(u_1) = \varphi'(u_2)$ and $\varphi'(v_1) = \varphi'(v_2)$. It follows that $w_1 \equiv w_2 \pmod 3$, and thus $w_1 \equiv w/2 \pmod 3$. Furthermore, by the construction of φ, the paths P_1 and P_2 can be partitioned into at most $6b + kc$ subpaths on which φ uses at most two colors, and thus $|w| \leq 2\mu = 2|R|$. Since R has even length and w is divisible by 4, Lemma 3.1 implies that we can extend φ to R so that $\delta_\varphi(R) = -w/2$.

Note that $\omega_{\varphi'}(\Gamma) = 0$: if Σ is orientable, this is the case since the sum of the winding numbers of the boundary cycles is 0 and for all other edges $e \in E(H)$, the contributions of the two edges of $\theta^{-1}(e)$ to the winding number of Γ cancel each other. If Σ is non-orientable, then this is the case by the choice of the coloring of R. We claim that φ' extends to a 3-coloring of G', and thus the corresponding 3-coloring of G extends ψ.

Suppose for a contradiction that φ' does not extend to a 3-coloring of G'. By Lemma 2.1, there exists a spoke P of Γ and its base Q such that $|P| < |\delta_{\varphi'}(Q)|$. By the construction of φ', we have $|\delta_{\varphi'}(Q)| \leq 4\mu$, and thus $|P| < 4\mu$. Let s and t be the endpoints of P. Let F be the subgraph of H formed by the edges $e \in E(\theta(Q))$ such that either e is a part of a boundary cycle, or exactly one of the two edges of $\theta^{-1}(e)$ belongs to Q. Note that F is obtained from the walk obtained as the image of Q by removing edges that appear twice in the walk. If $\theta(s) \neq \theta(t)$, then $\theta(s)$ and $\theta(t)$ have odd degree in F and all other vertices of F have even degree. Consequently, F contains a path Q' joining $\theta(s)$ with $\theta(t)$.

For any path $S \in \mathcal{P}$, we have $|E(S \cap Q')| \leq |P|$, since otherwise $H - (S \cap Q') + \theta(P)$ has fewer edges than H and it has only one face homeomorphic to a disk (since only one of the paths $\theta^{-1}(S \cap Q')$ belongs to Q), contrary to the choice of H. Thus, at most $3b|P| < 12b\mu \leq \beta$ edges of Q' do not belong to the boundary of Σ. We conclude that either Q' is a subpath of a path in $\mathcal{P}$, or Q' is disjoint from M.

In the former case, let $Q_1 = Q'$. In the latter case, either $Q' \subseteq L$, or there exists a boundary cycle C such that $Q' \subset C + L$. If no edge of Q' belongs to a boundary cycle, then let $Q_1 = Q'$. If some edge of Q' belongs to the boundary cycle C, then $Q' \cup \theta(P)$ contains a spoke of C, and since every essential subgraph of G has at least ρ edges, the spoke has a base. In this case, we let Q_1 be the union of the base with $Q' \cap L$.

Let $K = Q_1 + \theta(P)$. Note that K is contractible (when no edge of Q' belongs to a boundary cycle, this follows by the assumptions of the lemma, since $|K| < \rho$). Let $\Lambda' \subset \Sigma$ be the open disk bounded by K. Note that $\theta^{-1}(\Lambda')$ is one of the two faces of $\Gamma + P$, and thus $|\omega_\varphi(Q_1)| = |\omega_{\varphi'}(Q)| > |P|$. This implies that $|Q_1| > |P|$, and thus Q_1 is not a subset of a path in $\mathcal{P}$, by the minimality of H. Since G is a quadrangulation, $|Q_1|$ and $|P|$ have the same parity, and because $\omega_\varphi(Q_1)$ and $|Q_1|$ have the same parity, we have $|\omega_\varphi(Q_1)| \geq |P| + 2$. Therefore, $|\omega_\varphi(Q_1)| > 2$, and thus Q_1 is not a subpath of a single component of L and it must intersect the boundary cycle C. Let $Z_1, \ldots, Z_m \in \mathcal{Q}$ be the paths intersected by Q_1 in order. Since $|\omega_\varphi(Q_1)| > 2$, the construction of φ and the assumption that ψ is (d,k)-tame imply that $m \geq 3$. However, then $|Q_1 \cap C| > |Z_2| \geq d = \beta + 4\mu \geq |E(Q_1) \setminus E(C)|$, which contradicts the assumption that G is boundary-linked. □

Next, we need means to obtain a tame coloring.

Lemma 3.3. *Let $d \geq 1$ be an integer and let G be a quadrangulation of a cylinder Σ with boundary cycles B_1 and B_2, such that every non-contractible cycle has length at least $|B_1| = k$. Let ψ be a 3-coloring of B_1. If the distance between every non-contractible cycle of length less than $2k(d+1)$ and B_2 is at least $3k+4$, then ψ extends to a 3-coloring of G which is (k,d)-tame on B_2.*

Proof. First, note that since G is a quadrangulation, then for every $X \subseteq V(G)$ that separates B_1 from B_2, there exists a non-contractible cycle $C(X)$ such that $V(C(X)) \setminus X$ is an independent set in $C(X)$, and in particular, $|C(X)| \leq 2|X|$. For $i \geq 0$, let M_i be the set of vertices at distance exactly i from B_2, and let $C_i = C(M_{3i+2})$. Note that the cycles B_2, C_0, C_1, $\ldots$, are pairwise vertex-disjoint.

Let Z be the set of vertices of G that are not separated by C_k from B_2, including $V(C_k)$. If $X \subseteq Z$ separates $V(C_k)$ from $V(B_2)$, then $C(X)$ either intersects $V(C_k)$ or separates $V(C_k)$ from $V(B_2)$, and thus the distance between $C(X)$ and B_2 is at most $3k+3$. By the assumptions, the length of $C(X)$ is at least $2k(d+1)$, and thus $|X| \geq k(d+1)$. By Menger's theorem, there exist pairwise vertex-disjoint paths $P_1, \ldots, P_{k(d+1)}$ from $V(C_k)$ to $V(B_2)$, which intersect B_2 and C_k only in their endpoints. For $1 \leq i \leq k(d+1)$, let $\Delta_i \subset \Sigma$ be the closed disk bounded by paths P_i and P_{i+1} (where $P_{k(d+1)+1} = P_1$) and by subpaths of B_2 and C_k.

Let $\Sigma^\star$ be the sphere obtained from Σ by patching the cuffs with disks, and let $G^\star$ be the dual of G in its embedding in $\Sigma^\star$. Let b_1 and b_2 be the vertices of $G^\star$ corresponding to the faces of G bounded by B_1 and B_2, respectively. For $1 \leq i \leq k$, observe that there exists a path Q_i of $G^\star$ starting in b_2 and ending in an edge whose dual belongs to C_k, such that all but the first and the last edge of Q_i is contained in $\Delta_{i(d+1)}$. Similarly, there exists a cycle $K_i \subseteq G^\star$ such that the corresponding curve in Σ is non-contractible and contained in the subcylinder bounded by C_{i-1} and C_i. Let T be the set of size k consisting of the edges of the paths $Q_1, \ldots, Q_k$ whose dual belongs to B_2, and let R be the set of edges of the dual of B_2 that do not belong to T.

We claim that $G^\star$ contains k pairwise edge-disjoint paths $F_1^\star, \ldots, F_k^\star$ from b_1 to T that are disjoint from R. By Menger's theorem, it suffices to show that for every $Y \subseteq E(G^\star)$ of size less than k, the graph $G^\star - (Y \cup R)$ contains a path from b_1 to b_2. This is the case: since every non-contractible cycle of G has length at least k, Menger's theorem implies that $G^\star$ contains at least k pairwise edge-disjoint paths from b_1 to $V(K_1)$, and at least one of them is disjoint from Y; let us denote this path by A. Similarly, for some $1 \leq i, j \leq k$, the cycle K_i and the path Q_j are disjoint from Y. Then, $(A - R) + K_i + Q_j$ is a connected subgraph of $G^\star - (Y \cup R)$ containing both b_1 and b_2.

Note that we can choose the paths $F_1^\star, \ldots, F_k^\star$ so that they do not cross each other. For $1 \leq i \leq k$, let F_i denote the set of duals of the edges of $F_i^\star$. Let $B_1 = v_1 v_2 \ldots v_k$, where the edge $v_i v_{i+1}$ belongs to F_i for $1 \leq i \leq k$ (where $v_{k+1} = v_1$). If k is even, then let $\iota : V(G) \to \{1, 2\}$ be a 2-coloring of G. If k is odd, then let ι be a 2-coloring of $G - F_k$ such that $\iota(u) = \iota(v)$ for each $uv \in F_k$. In both cases, choose ι so that $\iota(v_1) = 1$. Let $f : \{1, \ldots, k\} \times \{1, 2\} \to \{1, 2, 3\}$ be defined by $f(i, c) = \psi(v_i)$ if i and c have the same parity and by $f(i, c) = \psi(v_{i-1})$ otherwise, where $v_0 = v_k$. Let $\varphi : V(G) \to \{1, 2, 3\}$ be defined as follows. For each $v \in V(G)$, there exists unique $i(v) \in \{1, \ldots, k\}$ such that the face of $G^\star$ dual to v is drawn in the region bounded by $F_{i(v)}^\star$ and $F_{i(v)+1}^\star$. We set $\varphi(v) = f(i(v), \iota(v))$.

Consider an edge $uv \in E(G)$. If $i(u) = i(v)$, then $\varphi(u) \neq \varphi(v)$, since $\iota(u) \neq \iota(v)$ and $f(i, 1) \neq f(i, 2)$ for every i. Thus, we can assume that either $i(u) = i(v) + 1$, or $i(u) = k$ and $i(v) = 1$. Observe that in both cases, $i(u) - i(v)$ and $\iota(u) - \iota(v)$ have the same parity, and thus either $\varphi(u) = \psi(v_{i(u)})$ and $\varphi(v) = \psi(v_{i(v)})$, or $\varphi(u) = \psi(v_{i(u)-1})$ and $\varphi(v) = \psi(v_{i(v)-1})$. Therefore, $\varphi(u) \neq \varphi(v)$, and thus φ is a 3-coloring of G.

Note that $B_2 \setminus (F_1 \cup \ldots \cup F_k)$ consists of k paths of length at least d, and φ uses only two colors on each of the paths. Furthermore, whenever P is the union of two such consecutive paths and the edge of $F_1 \cup \ldots \cup F_k$ between them, there exists a color class of φ whose complement is an independent set in P.

This implies that φ is (k, d)-tame. $\square$

Finally, we need to combine Lemmas 3.2 and 3.3.

Proof of Theorem 1.7. By Grötzsch's theorem, Corollary 2.2 and by Lemma 4.5 of [4], we can assume that Σ either has positive genus or at least three cuffs. Let d and ρ be the constants of Lemma 3.2 applied for Σ and k. Let $\sigma = \max(\rho, 2k(d+1))$ and $\nu = \rho + 2\sigma + 6k + 10$. We already argued in the introduction (Corollary 1.5 and Observation 1.6) that the winding number constraint is necessary for the existence of an extension of ψ. Let us now prove that it is sufficient.

Let $B_1, \ldots, B_m$ be the boundary cycles of G. For $1 \le i \le m$, let C_i be a cycle of length at most σ surrounding the cuff of B_i such that the part Σ_i of Σ between B_i and C_i is as large as possible (such a choice is possible, since B_i satisfies the requirements). Let M_i be the set of vertices of G contained in $\Sigma \setminus \Sigma_i$ such that the distance between each vertex of M_i and the cycle C_i in G is either $3k+4$ or $3k+5$, and let M'_i be the set of vertices of G contained in $\Sigma \setminus \Sigma_i$ such that the the distance between each vertex of M'_i and the cycle C_i in G is at most $3k+5$. For every $v \in M'_i$, let P_v be a path of length at most $3k+5$ between v and a vertex of C_i. Consider any cycle $K \subseteq G[M'_i]$. For every edge $uv \in E(K)$, all cycles in $H_{uv} = C_i + P_u + P_v + uv$ are either contractible or surround B_i, since H_{uv} is a connected subgraph of G with less than ν edges and thus cannot contain a connected essential subgraph. We conclude that K is either contractible or surrounds B_i. Since G contains a non-contractible cycle that does not surround B_i, it follows that $G[M_i]$ contains a cycle K_i that surrounds B_i. Let Σ'_i be the subcylinder of Σ between B_i and K_i. Observe that for $1 \le i < j \le m$, Σ'_i and Σ'_j are disjoint, as otherwise G would contain a connected essential subgraph with less than ν edges. Let G_i be the subgraph of G drawn in Σ'_i, and note that G_i satisfies the assumptions of Lemma 3.3. Hence, ψ extends to a 3-coloring φ_1 of $\bigcup_{i=1}^m G_i$ such that the restriction ψ' of φ_1 to $K_1 \cup \ldots K_m$ is (k, d)-tame on each of the cycles.

Let Σ' be the closure of $\Sigma \setminus \bigcup_{i=1}^m \Sigma'_i$, and let G' be the subgraph of G drawn in Σ'. Every cycle in G' that surrounds a cuff has length at least ρ by the choice of $C_1, \ldots, C_m$. Every connected essential subgraph H of G' corresponds to a connected essential subgraph of G with at most $|E(H)| + 2\sigma + 6k + 10$ edges, and thus H has at least $\nu - (2\sigma + 6k + 10) \ge \rho$ edges. By Lemma 3.2, ψ' extends to a 3-coloring φ_2 of G'. Therefore, ψ extends to a 3-coloring $\varphi_1 \cup \varphi_2$ of G. $\square$

4 The algorithm

Note that the proof of Lemma 2.1 gives an algorithm to decide whether the precoloring of a boundary of a disk extends to a 3-coloring of a quadrangulation of the disk (and to find such a coloring if it exists), by reducing the problem to finding the maximum number of edge-disjoint paths between prescribed vertices s and t (and by turning the resulting Z_3-flow to a 3-coloring). This corresponds to a network flow problem, which can be solved in linear time using Ford-Fulkerson algorithm when the degrees of s and t are bounded by a constant. Hence, we have the following.

Observation 4.1. *Theorem 1.3 is true if Σ is the sphere or the disk.*

For quadrangulations of the cylinder, we proved a result similar to Corollary 2.2 in the previous paper of the series.

Lemma 4.2 (Dvořák et al. [4], Corollary 4.6)**.** *For all positive integers d_1 and d_2, there exists a linear-time algorithm as follows. Let G be a boundary-linked quadrangulation of the cylinder with boundary cycles B_1 and B_2 such that $|B_1| = d_1$, $|B_2| = d_2$ and the distance between B_1 and B_2 is at least $d_1 + d_2$. Let ψ be a 3-coloring of $B_1 \cup B_2$ satisfying the winding number constraint. Then the algorithm returns a 3-coloring of G that extends ψ.*

Next, we need an algorithm to split a graph embedded in a cylinder to smaller subgraphs.

Lemma 4.3. *Let d be a positive integer. There exists a linear-time algorithm that, given a graph G that is 2-cell embedded in the cylinder Σ with boundary cycles B_1 and B_2 of length at most d, returns a sequence C_0, C_1, ..., C_m of non-contractible cycles of G of length at most d such that*

- *$C_0 = B_1$ and $C_m = B_1$,*
- *for $0 \le i < m$, the cycle C_i is contained in the part of Σ between B_1 and C_{i+1}, and*
- *either C_i intersects C_{i+1}, or the subcylinder of Σ between C_i and C_{i+1} contains no non-contractible cycle of length at most d distinct from C_i and C_{i+1}.*

Proof. For $2 \le k \le d+1$, we are going to construct an algorithm A_k with the same specification as in the statement of the lemma, under the additional assumption that every non-contractible cycle of length less than k shares an edge with one of the boundary cycles. Note that for $k = 2$, the assumption is void, and thus this will give a proof of Lemma 4.3.

We proceed by induction on decreasing k. First, let us assume that $k \le d$ and that the algorithm A_{k+1} exists. Let Σ' be the sphere obtained from Σ by patching the holes, and let s and t be the faces of G in its embedding in Σ' corresponding to the boundary cycles of G. Let $G^\star$ be the dual of G in its embedding in Σ', and let $s^\star$ and $t^\star$ be the vertices of $G^\star$ dual to s and t. Let F_0 consist of all edges of G incident with boundary cycles, and let $F_0^\star$ be the set of their duals. Consider a maximum flow from $s^\star$ to $t^\star$ in $G^\star/F_0^\star$ (where all edges have capacity 1).

The size of the flow is equal to the length of the shortest non-contractible cycle in $G - F_0$. By the assumption that every non-contractible cycle of length less than k shares an edge with one of the boundary cycles, the flow has size at least k. If the size of the flow is at least $k+1$, then the claim follows by applying the algorithm A_{k+1}. Therefore, assume that the maximum flow has size exactly k. Then, there exists a unique non-contractible k-cycle Q_1 in $G - F_0$ which is nearest to $s^\star$, corresponding to the cut of size k in $G^\star/F_0^\star$ bounding the set of vertices that can be reached from $s^\star$ by augmenting paths. Let F_1 consist of F_0 and all edges of $G - F_0$ drawn in the closed disk in Σ' bounded by Q_1 that contains s. Similarly, we find the non-contractible k-cycle Q_2 in $G - F_1$ which is nearest to $s^\star$, and so on, until no such k-cycle exists. Hence, we obtain a sequence of pairwise edge-disjoint non-contractible cycles $Q_0, Q_1, \ldots, Q_m$, where Q_0 and Q_m are the boundary cycles of G, such that for $0 \le i \le m-1$, if Q_i and Q_{i+1} are vertex-disjoint, then every non-contractible k-cycle of G drawn in the cylinder between Q_i and Q_{i+1} shares an edge with $Q_i \cup Q_{i+1}$. Note that we can inherit the flow in $G^\star/F_i^\star$ from $G^\star/F_{i-1}$, and thus in order to find the cycle Q_{i+1}, the algorithm visits only the edges in $F_{i+1} \setminus F_i$, for $0 \le i \le m-1$. Consequently we can find this sequence of cycles in linear time.

For $0 \le i \le m-1$, if Q_i and Q_{i+1} are not vertex-disjoint, then let S_i be the sequence consisting only of Q_i. If Q_i and Q_{i+1} are vertex-disjoint, then let S_i be the sequence obtained by applying algorithm A_{k+1} on the subgraph of G drawn between Q_i and Q_{i+1} (inclusive) except for its last element Q_{i+1}. We return the concatenation of the sequences $S_0, S_1, \ldots, S_{m-1}$, and the singleton B_2.

It remains to consider the case that $k = d+1$. Using the maximum flow algorithm as before, we find non-contractible cycles B_1' and B_2' of length at most d such that B_i and B_i' share an edge for $i \in \{1,2\}$ and such that no non-contractible cycle of length at most d other than B_1' and B_2' is drawn between B_1' and B_2'. Let S_1 be the sequence B_1, B_1' if $B_1 \ne B_1'$, and the singleton sequence B_1 otherwise. Let S_2 be the sequence B_2', B_2 if $B_2 \ne B_2'$, and the singleton sequence B_2 otherwise. If B_1' intersects B_2, then return the sequence B_1, B_1', B_2. Otherwise, return the concatenation of sequences S_1 and S_2. □

Using these tools, it is easy to deal with the cylinder case of Theorem 1.3.

Lemma 4.4. *Let d be a positive integer. There exists a linear-time algorithm that, given a quadrangulation G of the cylinder Σ with boundary cycles of length at most d and their precoloring ψ, decides whether ψ extends to a 3-coloring of G. If such a 3-coloring extending ψ exists, the algorithm outputs one.*

Proof. Let $C_1, \ldots, C_m$ be the sequence of cycles obtained by applying Lemma 4.3. For $1 \le i < j \le m$, let $G_{i,j}$ denote the subgraph of G drawn between C_i and C_j.

For $i = 1, \ldots, m$, let Ψ_i denote the set of all 3-colorings of $C_1 \cup C_i$ that extends to a 3-coloring of $G_{1,i}$. We determine the sets $\Psi_1, \ldots, \Psi_m$ by dynamic programming, and test whether ψ belongs to Ψ_m.

Clearly, Ψ_1 consists of all 3-colorings of the cycle C_1. Suppose that $i > 1$ and that we already determined the set Ψ_i. To compute Ψ_{i+1}, it suffices to determine the set Ψ_i' of all 3-colorings of $C_{i-1} \cup C_i$ that extend to the subgraph $G_{i-1,i}$. If the distance between C_{i-1} and C_i is less than $|C_1| + |C_2|$, then let

P be a shortest path between C_{i-1} and C_i. By using Observation 4.1, we can determine all 3-colorings of $C_{i-1} \cup P \cup C_i$ that extend to a 3-coloring of $G_{i-1,i}$, and thus also the set Ψ'_i. If the distance between C_{i-1} and C_i is at least $|C_1| + |C_2|$, then Ψ'_i consist of all 3-colorings of C_{i-1} and C_i that satisfy the winding number constraint, by Lemma 4.2.

Note that for each element ψ' of Ψ_i, we can also keep track of a 3-coloring of $G_{1,i}$ whose restriction to $C_1 \cup C_i$ is equal to ψ'_i. Hence, we can also return the 3-coloring of G whose restriction to $B_1 \cup B_2$ is equal to ψ, if such a coloring exists. □

The case of a general surface Σ is somewhat more involved. We need to specify how a graph G embedded in Σ is represented. Let g be the Euler genus of Σ and c the number of cuffs of Σ. We use the following variant of the polygonal representation. Let H be a graph drawn in Σ such that

- the boundary of every cuff traces a cycle in H,
- every edge H is either equal to an edge of G, or intersects G only in vertices,
- H has exactly one face and this face is homeomorphic to an open disk Λ, and
- H has at most $2(g + c - 1)$ vertices of degree three.

The graph G embedded in Σ is represented by a graph G' drawn in a closed disk Δ, such that there exists a continuous surjection $\theta : \Delta \to \Sigma$ satisfying

- $G' = \theta^{-1}(G)$,
- the restriction of θ to the interior of Δ is a homeomorphism to Λ, and
- θ maps the boundary of Δ to the boundary walk of the face Λ of H.

Note that there exist pairwise internally disjoint closed intervals A_1, B_1, A_2, B_2, ..., A_p, B_p for some $p \le 3(g + c) + 1$ such that the restriction of θ to each of them is injective, $\theta(A_i) = \theta(B_i)$ for $1 \le i \le p$, and $\bigcup_{i=1}^{p} \theta(A_i)$ is the subgraph of H consisting of edges not contained in any of the boundary cycles of G. Hence, the surface Σ is obtained from the disk Δ by identifying A_i with B_i for $1 \le i \le p$, in the direction prescribed by θ. We call this representation of an embedded graph a *normal representation*. Let us remark that a normal representation can be obtained from any other common representation of an embedded graph in linear time.

In [2], we designed a dynamic data structucture for representing first-order properties in sparse graphs. Here, the following special case will be important.

Lemma 4.5 (Dvořák et al. [2], Theorem 5.2)**.** *For every $d \ge 0$, there exists a data structure representing a planar graph G and a weight function $E(G) \to \{0, 1, \infty\}$, supporting the following operations in constant time (depending only on d):*

- *Removal of an edge or an isolated vertex.*
- *Changing the weight of any edge.*
- *For any vertices $u, v \in V(G)$ and integers $t, w \leq d$, deciding whether there exists a path between u and v with at most t edges and with total weight at most w, and finding such a path if it exists.*

The data structure can be initialized in $O(|V(G)|)$ time.

We use it to design a similar data structure for representing embedded graphs (inspired by the algorithm of Cabello and Mohar [1]).

Lemma 4.6. *For any integer $d \geq 0$ and every surface Σ, there exists a data structure as follows. Let $G_0 \cup F$ be a graph embedded in Σ such that F is a star forest. The data structure represents a graph G obtained from G_0 by contracting some edges of F and by removing vertices and edges. The data structure supports the following operations in amortized constant time (depending only on d and Σ):*

(a) Removal of an edge or an isolated vertex.

(b) Contraction of an edge of F.

(c) For any vertex $v \in V(G)$, deciding whether there exists a closed walk W of length at most d with $v \in V(W)$ such that W is not null-homotopic in Σ even after patching a single cuff with a disk, and finding such a walk if that is the case.

(d) For any vertex $v \in V(G)$ and any set D of cuffs of Σ, letting Σ' be the surface obtained from Σ by patching all the cuffs in D and letting $\Lambda \subseteq \Sigma'$ be an open disk containing all the patches, deciding whether there exists a closed walk W in G of length at most d such that W contains v and is homotopically equivalent (in Σ) to the boundary of Λ, and finding such a walk if that is the case.

Given a normal representation of G, the data structure can be initialized in $O(|V(G)|)$ time.

Proof. Let G' be the graph in a disk Δ and $\theta : \Delta \to \Sigma$ the surjection that form the normal representation of $G_0 \cup F$, and let A_1, B_1, $\ldots$, A_p, B_p be the intervals in the boundary of Δ as in the definition of the normal representation. Consider any sequence s of symbols from $\{a_1, b_1, \ldots, a_p, b_p\}$ of length at most d (including the empty one). We say that s is valid if no appearance of a_i is adjacent to an appearance of b_i in s. For every valid seqence s, let $(G^s, \Delta^s, \theta^s)$ be a disjoint copy of (G', Δ, θ), and let A_i^s and B_i^s be the corresponding intervals in the boundary of Δ^s. Glue the copies in a tree-like fashion as follows. For every non-empty valid sequence $s = s'a_i$, identify all points $x \in B_i^{s'}$ and $y \in A_i^s$ such that $\theta^{s'}(x) = \theta^s(y)$. Similarly, for every non-empty valid sequence $s = s'b_i$, identify all points $x \in A_i^{s'}$ and $y \in B_i^s$ such that $\theta^{s'}(x) = \theta^s(y)$.

In this way, we obtained a disk Δ_+ with a graph G_+ and a surjection $\theta_+ : \Delta_+ \to \Sigma$ such that $\theta_+(G_+)$ covers $G_0 \cup F$ several times. Note that for each $v \in V(G_0 \cup F)$, the size of $\theta_+^{-1}(v)$ is bounded by a function of d, the genus of Σ and the number of cuffs of Σ, which is a constant. Observe that for every walk W in $G_0 \cup F$ of length at most d, there exists a walk W_+ of the same length in G_+ such that $\theta_+(W_+) = W$. Furthermore, if W is closed, then its homotopy class is uniquely determined by valid sequences s_1 and s_2 such that the endpoints of W_+ lie in Δ^{s_1} and Δ^{s_2}. Hence, to find a closed walk W in $G_0 \cup F$ of the prescribed homotopy class such that W contains v and has length at most d, it is sufficient to try all combinations of s_1 and s_2 which correspond to this homotopy class and to test whether there exist vertices in $\Delta^{s_1} \cap \theta_+^{-1}(v)$ and $\Delta^{s_2} \cap \theta_+^{-1}(v)$ joined by a walk of length at most d.

To implement the data structure of Lemma 4.6, we use the data structure of Lemma 4.5 to represent G_+ with edge weights set to 1 for edges corresponding to edges of G and to ∞ for those corresponding to edges of F. Contraction of an edge of F is realized by changing the weight of the corresponding (constantly many) edges to 0. Removal of an edge or a vertex from G results in a removal of a constant number of edges or vertices from G_+ (if the vertex was obtained by contracting some edges in F, we may need to remove more vertices from the data structure, but this is amortized to the contraction operation). Hence we can update the data structure in constant time.

Since F is a star forest, the operations (d) and (e) can be implemented as described in the previous paragraph by testing all appropriate homotopy classes in constant time, using the operation of the data structure of Lemma 4.5 to find paths of length at most $3d + 2$ and weight at most d. □

The following lemma enables us to restrict our attention to boundary-linked quadrangulations without small essential subgraphs. We say that a surface Σ' is *at most as complex as* Σ if Σ' has smaller genus then Σ, or Σ' has the same genus and fewer cuffs than Σ, or Σ' is homeomorphic to Σ. For a graph G embedded in Σ, let $b(G)$ denote the multiset of the lengths of the boundary cycles of G. For two multisets S, T of integers such that $|S| = |T| = m$, we say that S *dominates* T if there exists an ordering $s_1, \ldots, s_m$ of the elements of S and an ordering $t_1, \ldots, t_m$ of the elements of T such that $s_i \geq t_i$ for $i = 1, \ldots, m$. We say that S *strictly dominates* T if S dominates T and $S \neq T$.

Lemma 4.7. *Let $\nu(\Sigma, k)$ be any function. For any surface Σ and an integer $k \geq 0$, there exists a constant σ and a linear-time algorithm as follows. Let G be a graph 2-cell embedded in Σ with boundary cycles $B_1, \ldots, B_c$ of total length at most k. The algorithm returns a subgraph H of G with at most σ vertices such that $B_1 \cup \ldots \cup B_c \subseteq H$ and for each face h of H, G_h (in its embedding in Σ_h) does not contain any connected essential subgraph with fewer than $\nu(\Sigma_h, k_h)$ edges, where k_h is the sum of the lengths of the boundary cycles of G_h. In addition, if Σ_h is not the cylinder, then G_h is boundary-linked. Furthermore, Σ_h is at most as complex as Σ, and if Σ_h is homeomorphic to Σ, then $b(G)$ dominates $b(G_h)$.*

Proof. We proceed by induction on the complexity of the surface and on k, i.e., we assume that the algorithm exists for surfaces at most as complex as Σ that either are not homeomorphic to Σ, or the total length of their boundary cycles is less than k.

The claim is obvious if Σ is the sphere (we can take H to be empty). If Σ is the cylinder, we test whether the distance between B_1 and B_2 in G is at least $\nu(\Sigma, |B_1| + |B_2|)$. If so, then G does not contain any connected essential subgraph with fewer than $\nu(\Sigma, |B_1| + |B_2|)$ edges by Lemma 1.8, and we return $H = B_1 \cup B_2$. Otherwise, let P be the shortest path between B_1 and B_2, and let f be the unique face of $B_1 \cup B_2 \cup P$. We apply the algorithm from induction to the graph G_f embedded in a disk, and return the graph obtained from the result by identifying the two paths in its boundary corresponding to P.

Suppose that Σ is the disk. If G is boundary-linked, we can take $H = B_1$. Otherwise, by performing breadth-first search from every vertex of B_1, we can in linear time find a spoke P of B_1 such that both bases of P have length less than $|P|$. Let f_1 and f_2 be the faces of $B_1 \cup P$. Note that both G_{f_1} and G_{f_2} are embedded in disks with boundary cycles of length less than k, and thus by induction, they have subgraphs H_1 and H_2, respectively, satisfying the conclusions of Lemma 4.7. We set $H = H_1 \cup H_2$.

Hence, assume that Σ either has non-zero genus or at least three cuffs. We build the data structure of Lemma 4.6 (with F empty), and by querying all vertices if necessary, we either find a closed walk W satisfying one of the following conditions or decide that there is no such walk:

(i) W has length at most $2\nu(\Sigma, k) + 2k$ and it is not null-homotopic in Σ even after patching a single cuff with a disk, or

(ii) W is a cycle homotopically equivalent to a boundary cycle B_i for some $i \in \{1, \ldots, c\}$ and $|W| < |B_i|$.

Using Lemma 1.8, it is easy to see that that if no such walk exists, then G is boundary-linked and contains no essential subgraph with less than $\nu(\Sigma, k)$ edges; hence, we can set $H = B_1 \cup \ldots \cup B_c$.

Suppose that a walk W as in (i) exists. Then we can find a connected subgraph S of G with $E(S) \subseteq E(W)$ such that S satisfies one of the conclusions of Lemma 1.8. For each face f of $S' = S \cup B_1 \cup \ldots \cup B_c$, the surface Σ_f has either smaller genus than Σ, or the same genus and fewer cuffs. Similarly, if a walk W as in (ii) exists, then each face f of $S = W \cup B_1 \cup \ldots \cup B_c$ is either an open cylinder, or Σ_f is homeomorphic to Σ and $b(G)$ strictly dominates $b(G_f)$.

Hence, in both cases we can apply induction for G_f, obtaining its subgraph H_f. We let H be the union of the graphs H_f over all faces f of S'. □

To deal with the boundary-linked case without small essential subgraphs, we use the following shrinking lemma. Let G and F be graphs embedded in the same surface such that the embeddings only intersect in vertices. A *diagonal* of a 4-face $f = v_1v_2v_3v_4$ of G is an edge e of F joining either v_1 with v_3, or v_2 with v_4, and drawn inside f. By *contracting the diagonal* $e = v_iv_{i+2}$, we mean identifying the ends of e to a single vertex v (modifying the embeddings of G and F in the natural way) and suppressing the arising 2-faces $v_{i+1}v$ and $v_{i-1}v$ (where $v_0 = v_4$).

Lemma 4.8. *For any surface Σ other than the sphere with at most two cuffs, and for all integers $k, \nu \geq 0$, there exists a linear-time algorithm as follows. Let G be a boundary-linked quadrangulation of Σ with boundary cycles $B_1, \ldots, B_c$ of total length at most k, such that G does not contain any connected essential subgraph with fewer than ν edges. Let F be a star forest embedded in Σ so that the embeddings of F and G intersect only in vertices, such that each edge of F is a diagonal of a 4-face of G.*

The algorithm returns a boundary-linked quadrangulation G' of Σ and a forest F' embedded in Σ obtained from G and F, respectively, by contracting some of the diagonals of F, such that G' does not contain any connected essential subgraph with fewer than ν edges, and a surface $\Sigma' \subseteq \Sigma$ homeomorphic to Σ such that $G' \cap \Sigma'$ is a quadrangulation of Σ' with boundary cycles $B'_1, \ldots, B'_c$ of the same length as the corresponding boundary cycles of G, and one of the following holds.

- *$G' \cap \Sigma'$ contains a connected essential subgraph with at most $2\nu + 2k + 2$ edges, or*
- *$F' \cap \Sigma' = \emptyset$.*

Proof. We use the data structure of Lemma 4.6. Initially, set $\Sigma' = \Sigma$, $G' = G$ and $F' = F$. We process the diagonals of F one by one. If the currently processed diagonal uv does not lie in Σ', we ignore it. Otherwise, we contract uv in G' to a vertex w, obtaining a new graph G'' with forest F''. We use the data structure to test whether $G'' \cap \Sigma'$ contains a closed walk W' containing w, such that either W' has length at most $2\nu + 2k$ and it is not null-homotopic in Σ even after patching a single cuff with a disk, or W' is a cycle homotopically equivalent to a bountary cycle B_i for some $i \in \{1, \ldots, c\}$ and $|W'| < |B_i|$. If no such closed walk exists, observe that $G'' \cap \Sigma'$ is boundary linked and does not contain any connected essential subgraph with fewer than ν edges. In this case, we set $G' = G''$, $F' = F''$ and proceed with further diagonals of F.

If W' exists and is not null-homotopic in Σ even after patching a single cuff with a disk, then we set $G' = G''$, $F' = F''$, and the algorithm ends—the graph $G'' \cap \Sigma'$ contains a connected essential subgraph with at most $2\nu + 2k + 2$ edges.

Suppose now that W' exists and is homotopically equivalent to a boundary cycle B_i and $|W'| < |B_i|$. Since G'' is a quadrangulation, $|W|$ and $|B'_i|$ have the same parity. And since G' is boundary-linked, we conclude that $|W'| = |B_i| - 2$. Note that G' contains a cycle $W \neq B_i$ of length $|B_i|$ that is homotopically equivalent to B_i, such that the diagonal uv is contained in the part of Σ' between B_i and W. In this case, we alter Σ' by removing the cylinder between B_i and W, excluding the cycle W which becomes a new boundary cycle. (to reflect this in the data structure, it suffices to remove all the vertices and edges of G' that belong to the removed cylinder). We keep the current graph G' and forest F' and we proceed with processing futher diagonals of F.

Note that in the previous paragraph, the removed cylinder contains the diagonal uv. Hence, if the algorithm processes all diagonals of F, then $G' \cap \Sigma'$ contains no non-contracted diagonals of F as required by the second outcome of the lemma. $\square$

Next, we design an algorithm to get rid of short contractible separating cycles.

Lemma 4.9. *For any surface Σ, there exists a linear-time algorithm that for any graph G with a 2-cell embedding in Σ returns its subgraph H such that*

- *all boundary cycles of G belong to H,*
- *all contractible cycles in H of length at most 4 bound 2-cell faces, and*
- *all vertices and edges of G that do not belong to H are drawn in 2-cell (≤ 4)-faces of H.*

Proof. First, let us prepare a data structure similarly to Lemma 4.6. Let G' be the graph drawn in a disk Δ and $\theta : \Delta \to \Sigma$ the surjection that form the normal representation of G, and let A_1, B_1, …, A_p, B_p be the intervals in the boundary of Δ as in the definition of the normal representation. For every valid seqence s of length at most two, let $(G^s, \Delta^s, \theta^s)$ be a disjoint copy of (G', Δ, θ), and let A_i^s and B_i^s be the corresponding intervals in the boundary of Δ^s. Glue the copies in a tree-like fashion as in the proof of Lemma 4.6.

In this way, we obtained a graph G_0 drawn in a disk Δ_0 and a surjection $\theta_0 : \Delta_0 \to \Sigma$ such that $\theta_0(G_0)$ covers G $(4p^2+1)$ times. For every closed walk W_0 in G_0, the walk $\theta_0(W_0)$ in G is contractible. Conversely, for every contractible closed walk W in G of length at most 4, there exists a closed walk W_0 of length 4 in G_0 such that $W = \theta_0(W_0)$. Build the data structure of Lemma 4.5 for G_0 with $d = 3$, initally setting the weights of all edges to 1.

Now, given an edge $e = uv$ of G, we can determine whether e is contained in a non-facial contractible (≤ 4)-cycle in G in constant time as follows. A set X *breaks (≤ 4)-faces* at e if X contains exactly one edge of each (≤ 4)-face incident with e and no other edges, and $e \notin X$. Note that there are at most 9 sets that break (≤ 4)-faces at e, and given the embedding of G, they can be enumerated in constant time. Let $e_0 = u_0v_0$ be an edge of G_0 such that $\theta_0(e_0) = e$. For each set X that breaks (≤ 4)-faces at e, set the weights of edges in $\theta_0^{-1}(X \cup \{e\})$ to ∞, test whether G_0 contains a walk of length and weight at most 3 between u_0 to v_0, and restore the weights of all edges to 1. If such a walk is found, then its image together with the edge uv forms a non-facial contractible (≤ 4)-cycle in G. Otherwise, no such cycle exists.

Now, for each edge $e \in E(G)$, we try to find such a cycle C, and if it exists, we remove all the vertices and edges of G contained in the open disk bounded by C (and update the data structure accordingly). We repeat this procedure for each edge e until all contractible cycles of length at most 4 that contain e bound 2-cell faces. Note that we remove each vertex and edge at most once, hence the total time complexity of the algorithm is linear. $\square$

Finally, we need the following result.

Theorem 4.10 (Nešetřil and Ossona de Mendez [11]). *For every $g \geq 0$, there exists $\mu \geq 0$ and a linear-time algorithm that for a simple graph of Euler genus at most g finds a coloring of its vertices by at most μ colors such that the union of every two color classes induces a star forest.*

The algorithm of Theorem 1.3 is now straightforward.

Proof of Theorem 1.3. We proceed by induction on the complexity of the surface, i.e., we assume that the algorithm exists for quadrangulations of surfaces at most as complex as Σ that are not homeomorphic to Σ. By Observation 4.1 and Lemma 4.4, we can assume that Σ either has non-zero genus or at least three cuffs.

Let $\nu(\Sigma, k) \geq 5$ be function defined so that for each surface Σ and integer $k \geq 0$, Theorem 1.7 holds with $\nu := \nu(\Sigma, k)$. Let us apply the algorithm of Lemma 4.7 with this function ν; let H be the resulting subgraph of G. For each extension ψ' of ψ to H, we test whether ψ' extends to a 3-coloring of G_h for every face h of H. If that is the case, we also find the extensions, which together give a 3-coloring of G extending ψ. Hence, it suffices to discuss how to extend ψ' to G_h for a face h of G.

If Σ_h is not homeomorphic to Σ, we apply the algorithm which exists by induction. Hence, assume that Σ_h is homeomorphic to Σ, and thus Σ_h is not a cylinder and the sum of lengths of boundary cycles of G_h is at most k. Furthermore, G_h is boundary-linked and contains no essential subgraph with less than $\nu(\Sigma, k)$ edges. By Theorem 1.7, ψ' extends to G_h if and only if ψ' satisfies the winding number constraint, and this can be tested in linear time.

Suppose that ψ' satisfies the winding number constraint, and thus we need to find a 3-coloring of G_h that extends ψ'. We construct a sequence (G_0, Σ_0, ψ_0), (G_1, Σ_1, ψ_1), $\ldots$, (G_r, Σ_r, ψ_r), where for $0 \leq i \leq r$, G_i is a boundary-linked quadrangulation of Σ_i with no essential subgraph with less than $\nu(\Sigma, k)$ edges, Σ_i is homeomorphic to Σ and ψ_i is a 3-coloring of the boundary cycles of G_i, as follows. Let n_i denote the number of vertices of G_i that are not contained in its boundary cycles.

Let $G_0 = G_h$, $\Sigma_0 = \Sigma_h$ and let ψ_0 be the restriction of ψ' to the boundary cycles of G_h. Suppose that we already constructed G_i for some $i \geq 0$. To obtain G_{i+1}, first take its subgraph using Lemma 4.9 and suppress the resulting 2-faces. Let G_i' denote the resulting graph. Since G_i is boundary-linked, observe that G_i' contains no parallel edges and that G does not contain three distinct 4-faces $u_1v_1w_1x_1$, $u_2v_2w_2x_2$ and $u_3v_3w_3x_3$ such that $u_1 = u_2 = u_3$ and $w_1 = w_2 = w_3$. Let n_i' denote the number of vertices of G_i' that are not contained in its boundary cycles.

Next, let F_i' be a maximal graph embedded in Σ_i such that each edge of F_i' is a diagonal of a face of G_i' and F_i' has no parallel edges. Note that $|E(F_i')|$ is at least half the number of faces of G_i, and thus $|E(F_i')| \geq n_i'/2$. By Theorem 4.10, there exists a subgraph F_i of F_i' with at least $\frac{n_i'}{\mu^2}$ edges such that F_i is a star forest. Apply Lemma 4.8 to G_i' and F_i, let G_i'' and Σ_{i+1} be the resulting graph and surface, and let $G_{i+1} = G''_i \cap \Sigma_{i+1}$. Use Lemma 4.4 to find a 3-coloring ψ'_i of $G''_i \cap \overline{\Sigma \setminus \Sigma'}$ that extends ψ_i, and let ψ_{i+1} be the restriction of this coloring to the boundary cycles of G_{i+1}. If G_{i+1} contains a connected essential subgraph with at most $2\nu(\Sigma, k) + 2k + 2$ edges, then let $r := i + 1$, otherwise proceed with the construction.

Note that if $i < r - 1$, then G_{i+1} contains no non-contracted diagonals of F_i, and thus $n_{i+1} \le n'_i - |F_i| \le (1 - 1/\mu^2)n'_i \le (1 - 1/\mu^2)n_i$. Observe that $|V(G_i)| \le 3n_i$, since G_i is boundary-linked and does not contain an essential subgraph with at most two edges. Hence, each step of the construction has time complexity linear in n_i, and thus the total time complexity for the construction of the sequence is $O\left(\sum_{i=0}^{r-1} n_i\right) \le O\left(|V(G)| \sum_{i \ge 0} (1 - 1/\mu^2)^i\right) = O(|V(G)|)$.

Now, G_r contains a connected essential subgraph with at most $2\nu(\Sigma, k) + 2k + 2$ edges. Let H_r be the union of the boundary cycles of G_r with this essential subgraph, and observe that for each face h of H_r, the surface $\Sigma_{r,h}$ is at most as complex as Σ and not homeomorphic to Σ. Hence, we can find a 3-coloring of G_r extending ψ_r (which exists by Theorem 1.7) by trying all the possible extensions of ψ_r to H_r and for each of them applying induction to the subgraphs drawn in the faces of H_r.

Furthermore, for $0 \le i \le r - 1$, given a 3-coloring φ_{i+1} of G_{i+1} that extends ψ_{i+1}, we can obtain a coloring of G_i that extends ψ_i in linear time as follows. First, let $\varphi''_i = \psi'_i \cup \varphi_{i+1}$ be a 3-coloring of G''_i that extends ψ_i. Next, to all vertices that were identified to a single vertex w by contracting the diagonals in F_i give the same color as w, thus obtaining a 3-coloring φ'_i of G'_i that extends ψ_i. Finally, use the algorithm of Observation 4.1 to extend this coloring to the parts of G_i removed during the construction of G'_i.

Then, φ_0 is a 3-coloring of G_h which extends ψ', as required. □

5 Important subgraphs of embedded graphs

We need a stronger form of Theorem 1.2 which deals with graphs with precolored cycles. First, let us give several definitions. Let B be a subgraph of G. We say that G is *B-critical* if $G \ne B$ and for every proper subgraph G' of G such that $B \subseteq G'$, there exists a 3-coloring of B that extends to a 3-coloring of G', but not to a 3-coloring of G. Note that G is 4-critical if and only if G is $\emptyset$-critical.

Suppose that a graph G is embedded in a surface Σ so that every cuff of Σ traces a cycle in G. To each face f of G, we assign a weight $w_0(f)$ as follows. If f is homeomorphic to an open disk, then let

$$w_0(f) = \begin{cases} 0 & \text{if } |f| \le 4 \\ 4/4113 & \text{if } |f| = 5 \\ 72/4113 & \text{if } |f| = 6 \\ 540/4113 & \text{if } |f| = 7 \\ 2184/4113 & \text{if } |f| = 8 \\ |f| - 8 & \text{if } |f| \ge 9. \end{cases}$$

If f is not homeomorphic to an open disk, then let $w_0(f) = |f|$. For a surface Π of Euler genus g with c cuffs, let $s(\Pi) = 6c - 6$ if $g = 0$ and $c \leq 2$, and $s(\Pi) = 120g + 48c - 120$ otherwise. Recall that Σ_f was defined prior to Theorem 1.9. For a real number η and a face f of G, let $w_\eta(f) = w_0(f) + \eta s(\Sigma_f)$. Let

$$w_\eta(G) = \sum_{f \text{ face of } G} w_\eta(f).$$

Theorem 5.1 ([Dvořák et al. [3], Corollary 5.6)**.** *There exists a constant η such that the following holds. Let G be a triangle-free graph embedded in a surface Σ without non-contractible 4-cycles, so that every cuff of Σ traces a cycle in G, and let B be the union of boundary cycles of G. If G is B-critical, then $w_\eta(G) \leq w_\eta(B)$.*

By iterating Theorem 5.1, we obtain a variant of Theorem 1.9.

Lemma 5.2. *Let η be the constant of Theorem 5.1. Let G be a triangle-free graph embedded in a surface Σ without non-contractible 4-cycles, so that every cuff of Σ traces a cycle in G, and let B be the union of boundary cycles of G. There exists a subgraph H of G such that $B \subseteq H$, $w_\eta(H) \leq w_\eta(B)$ and for every face h of H, every 3-coloring of the boundary of h extends to a 3-coloring of G_h.*

Proof. Let H be a maximal subgraph of G such that $B \subseteq H$ and $w_\eta(H) \leq w_\eta(B)$. Consider any face h of H, and let B_h be the union of boundary cycles of G_h in its embedding in Σ_h. If there exists a 3-coloring of B_h that does not extend to a 3-coloring of G_h, then G_h contains a B_h-critical subgraph H_h. Let H'_h be the subgraph of G corresponding to H_h. By Theorem 5.1, we have $w_\eta(H_h) \leq w_\eta(B_h)$, and thus $w_\eta(H \cup H'_h) \leq w_\eta(H) \leq w_\eta(B)$. This contradicts the maximality of H. Therefore, H satisfies the conclusions of Lemma 5.2. □

Note that Lemma 5.2 almost implies Theorem 1.9, up to non-contractible 4-cycles and quadrangulations. The former are easy to deal with by cutting the surface, and we already analyzed quadrangulations in Lemma 4.7.

Proof of Theorem 1.9. We proceed by induction on the complexity of the surface and on k, i.e., we assume that the theorem holds for graphs embedded in surfaces at most as complex as Σ that either are not homeomorphic to Σ, or the total length of their boundary cycles is less than k.

Suppose first that

- G contains a connected essential subgraph K with at most 8 edges, or
- Σ is not the cylinder and G contains a boundary cycle B and a cycle K surrounding it such that $|K| < |B|$, or
- Σ is the cylinder and G contains a non-contractible cycle K shorter than both boundary cycles of G.

Then, we cut the surface along K, apply induction to the pieces and let H be the union of the resulting subgraphs. Hence, assume that G has no such subgraph.

In particular, every non-contractible 4-cycle surrounds a cuff bounded by a 4-cycle. If Σ is a cylinder and both of its boundary cycles B_1 and B_2 have length 4, then we let $H = B_1 \cup B_2$. The graph H clearly satisfies the conclusion of the theorem. Otherwise, let $B_1, \ldots, B_t$ be the boundary cycles of G of length 4, and for $1 \le i \le t$, let K_i be a 4-cycle surrounding B_i such that the subset Σ_i of Σ between B_i and K_i is maximal. Note that for $i \neq j$, we have $\Sigma_i \cap \Sigma_j = \emptyset$, since G does not contain an essential subgraph with at most 8 edges and Σ is not a cylinder with two boundary 4-cycles. Without loss of generality, we can assume that B_i and K_i are vertex-disjoint for $1 \le i \le t'$ and share a vertex v_i for $t' + 1 \le i \le t$. For $1 \le i \le t$, let v_i be an arbitary vertex of K_i. Let $\Sigma' = \overline{\Sigma \setminus \bigcup_{i=1}^{t'} \Sigma_i}$ and $G' = G \cap \Sigma'$. Note that Σ' is homeomorphic to Σ. Let G'' be obtained from G' by, for $1 \le i \le t$, splitting the vertex v_i into two vertices v_i' and v_i'' and adding a new vertex v_i''' adjacent to v_i' and v_i'' (drawn in the boundary of Σ'), so that G'' contains no non-contractible 4-cycles.

Let H_1 be the subgraph of G'' obtained using Lemma 5.2, and let H_2 be the subgraph of H_1 consisting of the boundary cycles of G'' and of all boundary walks of faces of H_1 of length greater than 4. Since $w_\eta(H_1) \le 5/4k + \eta s(\Sigma)$ and since each face of length greater than 4 is incident with at most $4113 w_\eta(f)$ edges, it follows that H_2 has at most $5/4k + 4113(5/4k + \eta s(\Sigma))$ edges. Furthermore, for each face h of H_2, either h is a face of H_1, and thus every precoloring of its boundary extends to a 3-coloring of G_h'', or all faces of H_1 contained in H have length 4. Since G'' does not contain non-contractible 4-cycles, and since every contractible 4-cycle bounds a face of G by the assumptions of Theorem 1.9, it follows that G_h'' is a quadrangulation in this case.

Let $\nu(\Sigma, k) \ge 5$ be function defined so that for each surface Σ and integer $k \ge 0$, Theorem 1.7 holds with $\nu := \nu(\Sigma, k)$. Let H_3 be obtained from H_2 as follows: for each face h of H_2 such that G_h'' is a quadrangulation, we apply Lemma 4.7 to G_h'' and replace the face h by the resulting subgraph. Hence, the size of H_3 is bounded by some constant depending only on Σ and k, and every face h of H_3 satisfies one of the following:

- h is a face of H_1, and thus h satisfies (a) of Theorem 1.9, or
- G_h is a quadrangulation and h is an open cylinder, and thus h satisfies (c) of Theorem 1.9, or
- G_h is a boundary-linked quadrangulation with no connected essential subgraph with fewer than $\nu(\Sigma_h, k_h)$ edges, where k_h is the sum of the lengths of the boundary cycles of G_h. In this case Theorem 1.7 implies that h satisfies (b) of Theorem 1.9.

Finally, let H be obtained from H_3 by contracting edges $v_i'v_i'''$ and $v_i'v_i'''$ and adding the cycle B_i for $1 \le i \le t'$. Note that the face of G'' incident with $v_i'v_i'''v_i''$ does not have length 4, and thus the face of H_3 incident with this path satisfies (a) of Theorem 1.9; this is not changed by eliminating the vertex v_i''' of degree two. Furthermore, the faces of H incident with $B_1, \ldots, B_{t'}$ satisfy (d) of Theorem 1.9. $\square$

Before we prove Corollary 1.10, we need some results on coloring planar graphs.

Theorem 5.3 (Gimbel and Thomassen [7]). *Let G be a triangle-free graph embededded in a disk Δ with a boundary cycle B of length at most 6. If some precoloring of B does not extend to a 3-coloring of G, then $|B| = 6$ and G has a subgraph that quadrangulates Δ.*

As a corollary, we obtain the following.

Lemma 5.4. *Let G be a connected triangle-free plane graph and let B be the boundary walk of its outer face. Then G has a 3-coloring ψ_1 with winding number $|\omega_{\psi_1}(B)| \le 1$. Furthermore, if G does not contain a subgraph Q such that $B \subseteq Q$ and all inner faces of Q have length 4, then G has a 3-coloring ψ_2 with $1 \le |\omega_{\psi_2}(B)| \le 2$.*

Proof. Let $B = b_1 b_2 \ldots b_k$. Let G' be the graph obtained from G by adding two cycles $v'_1 v'_2 \ldots v'_k$ and $v_1 v_2 \ldots v_k$ and edges $b_i v'_i$ and $v'_i v_i$ $(1 \le i \le k)$ and $v_1 v_{2+2i}$ $(1 \le i \le \lfloor k/2 \rfloor - 2)$. Let $B' = v_{2\lfloor k/2 \rfloor - 2} v_{2\lfloor k/2 \rfloor - 1} \ldots v_1$. Note that $4 \le |B'| \le 5$. By Grötzsch's Theorem, G' has a 3-coloring ψ. Note that $|\omega_\psi(B')| \le 1$ and by Corollary 1.5, $\omega_\psi(B) = \omega_\psi(B')$. Hence, we can choose ψ_1 as the restriction of ψ to G.

Now, suppose that G does not contain a subgraph Q as described in the statement of the lemma, and in particular $k \ge 5$. If k is odd, then $|\omega_\psi(B')| = 1$ and the claim of Lemma 5.4 follows by setting $\psi_2 = \psi_1$. Suppose that k is even. Let $G'' = G' - v_1 v_{k-2}$ and $B'' = v_{k-4} v_{k-3} \ldots v_1$. We have $|B''| = 6$ and G'' does not contain a quadrangulation, and thus by Theorem 5.3, it has a 3-coloring ψ such that $\psi(v_{k-4}) = \psi(v_{k-1}) = 1$, $\psi(v_{k-3}) = \psi(v_k) = 2$ and $\psi(v_{k-2}) = \psi(v_1) = 1$. Note that $\omega_\psi(B) = \omega_\psi(B') = 2$, hence we can set ψ_2 to be the restriction of ψ to G. $\square$

Now, we are ready to deal with the graphs of large edge-width.

Proof of Corollary 1.10. By Grötzsch's theorem, we can assume that Σ is not the sphere. Let $\gamma = \max(5, \beta + 1)$, where β is the constant of Theorem 1.9. We can assume that G is 4-critical, and thus by Theorem 5.3, every 4-cycle in G bounds a face. Let H be the subgraph of G as in Theorem 1.9. Since every cycle in H has length less than γ, it follows that all cycles in H are contractible. Therefore, H has exactly one face h such that Σ_h has non-zero genus. Let G_0 be the subgraph of G drawn in the closure of h, and let $G_1, \ldots, G_m$, be the subgraphs of G drawn in the components of $\Sigma \setminus h$. Observe that for $1 \le i \le m$, G_i is a plane graph with outer face bounded by a closed walk B_i, such that B_i is one of the facial walks of h.

The face h satisfies (a) or (b) of Theorem 1.9. If it satisfies (a), then we can 3-color each of the graphs $G_1, \ldots, G_k$ arbitrarily using Grötzsch's theorem and extend the coloring to G_0. Hence suppose that h satisfies (b), i.e., G_h is a quadrangulation and any precoloring of $B_1 \cup \ldots \cup B_m$ which satisfies the winding number constraint extends to a 3-coloring of G_0. Since the dual of G_h has even number of vertices of odd degree, it follows that even number of the boundary walks $B_1, \ldots, B_m$ has odd length.

If Σ is orientable, then 3-color $G_1, \ldots, G_m$ using Lemma 5.4 so that the winding number of each of the boundary walks is 0 or ± 1. Note that by permuting colors, a coloring with winding number 1 on some walk W can be transformed to a coloring with winding number -1 on W. Therefore, we can choose the colorings so that the sum of winding numbers of $B_1, \ldots, B_m$ is 0, and thus it satisfies the winding number constraint in G_0. Thus, we can extend it to a 3-coloring of G.

Suppose now that Σ is non-orientable. If G is a quadrangulation, then the assumptions of Corollary 1.10 imply that it satisfies the winding number constraint, and thus G is 3-colorable by Theorem 1.7. Hence, suppose that say G_1 has an inner face of length greater than 4. Again, choose the coloring ψ of $G_1 \cup \ldots \cup G_m$ so that the winding number of each of the boundary walks is 0 or ± 1 and the sum of the winding numbers of boundary walks is 0. If this coloring is parity-compliant in G_0, then we can extend it to a 3-coloring of G. Otherwise, we alter the coloring of G_1. If $|\omega_\psi(B_1)| = 1$, say $\omega_\psi(B_1) = -1$, we permute the colors in the coloring of G_1 so that $\omega_\psi(B_1) = 1$. If $\omega_\psi(B_1) = 0$, we use Lemma 5.4 to find a coloring of G_1 so that $\omega_\psi(B_1) = 2$. In both cases, we obtain a 3-coloring of $G_1 \cup \ldots \cup G_m$ such that the sum of winding numbers of boundary cycles is 2, and thus it is parity-compliant in G_0 and can be extended to a 3-coloring of G. □

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