DIRECTED TREE-WIDTH

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joint work with

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OUTLINE

- Tree-width and havens for undirected graphs
- Even directed circuits
- Packing directed circuits
- Path-width of directed graphs
- Tree-width of directed graphs
- Havens in directed graphs
- Algorithms

A tree-decomposition of a graph G is (T, W), where T is a tree and $W = (W_t : t \in V(T))$ satisfies

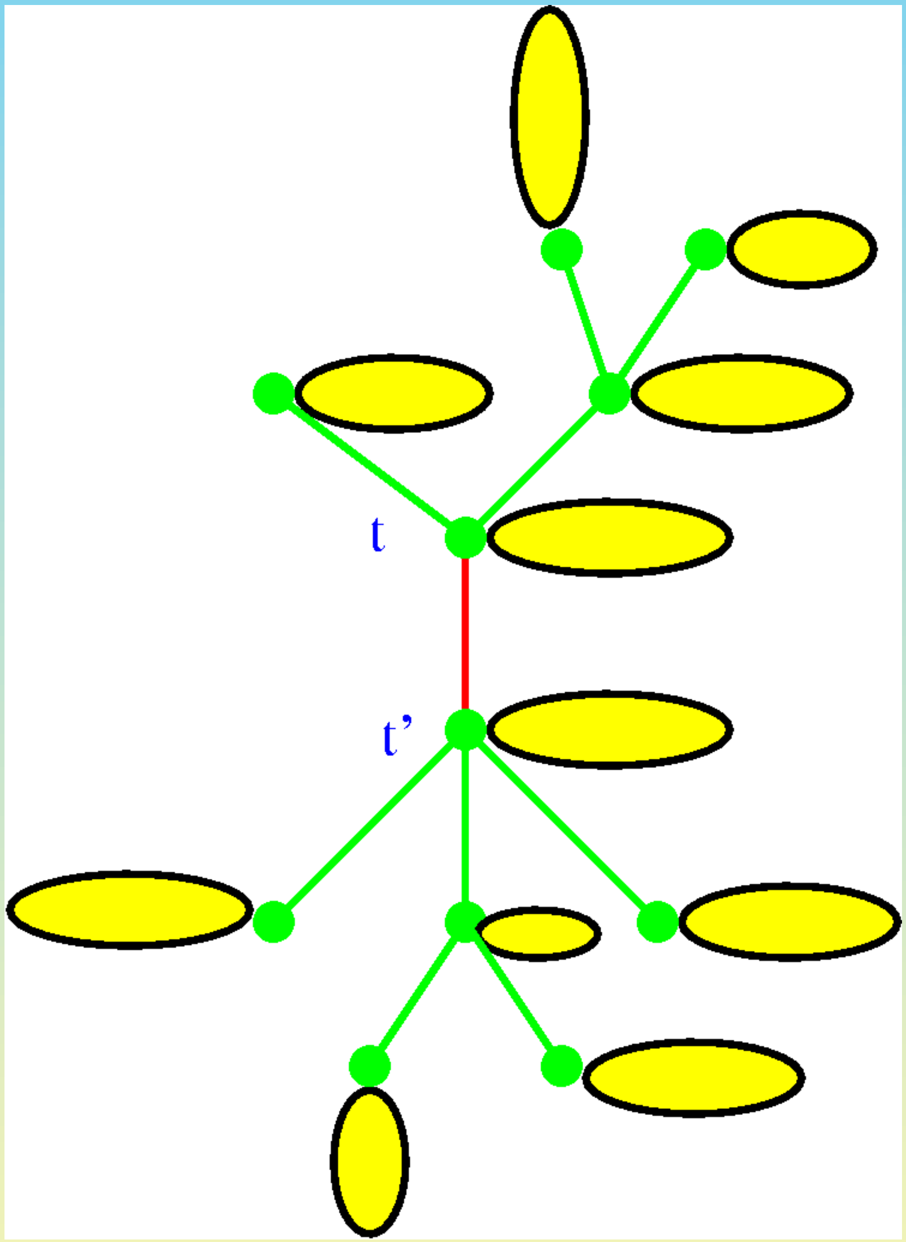
(T1) $\bigcup_{t \in V(T)} W_t = V(G)$,

(T2) if $t' \in T[t, t'']$, then $W_t \cap W_{t''} \subseteq W_{t'}$,

(T3) $\forall uv \in E(G) \exists t \in V(T)$ s.t. $u, v \in W_t$.

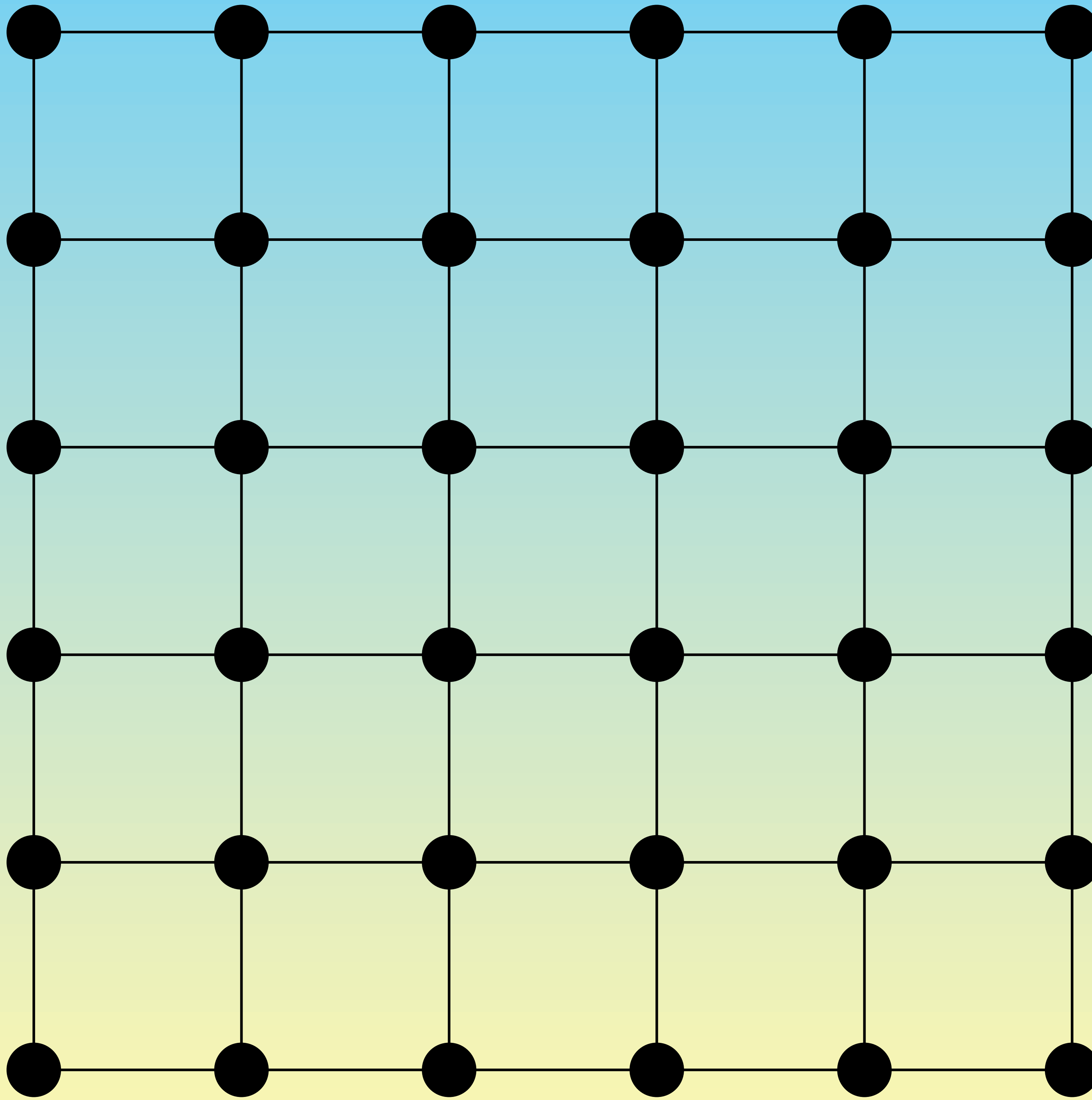
The width is $\max(|W_t| - 1 : t \in V(T))$.

The tree-width of G is the minimum width of a tree-decomposition of G.


t
t'

- $tw(G) \leq 1 \Leftrightarrow G$ is a forest
- $tw(G) \leq 2 \Leftrightarrow G$ is series-parallel
- $tw(G) \leq 3 \Leftrightarrow$ no minor isomorphic to:

  K_5, 5-prism, octahedron, V_8
- $tw(K_n) = n - 1$
- tree-width is minor-monotone
- The $k \times k$ grid has tree-width k



Consider all functions ϕ mapping graphs into integers such that

(1) $\phi(K_n) = n - 1$,

(2) G minor of $H \Rightarrow \phi(G) \leq \phi(H)$,

(3) If $G \cap H$ is a clique, then
$\phi(G \cup H) = \max\{\phi(G), \phi(H)\}$.

Order such functions by $\phi \leq \psi$ if $\phi(G) \leq \psi(G)$ for all G.

THEOREM (Halin) Tree-width is the maximum element in the above poset.

A haven β of order k in G assigns to every $X \in [V(G)]^{<k}$ the vertex-set of a component of $G \backslash X$ such that

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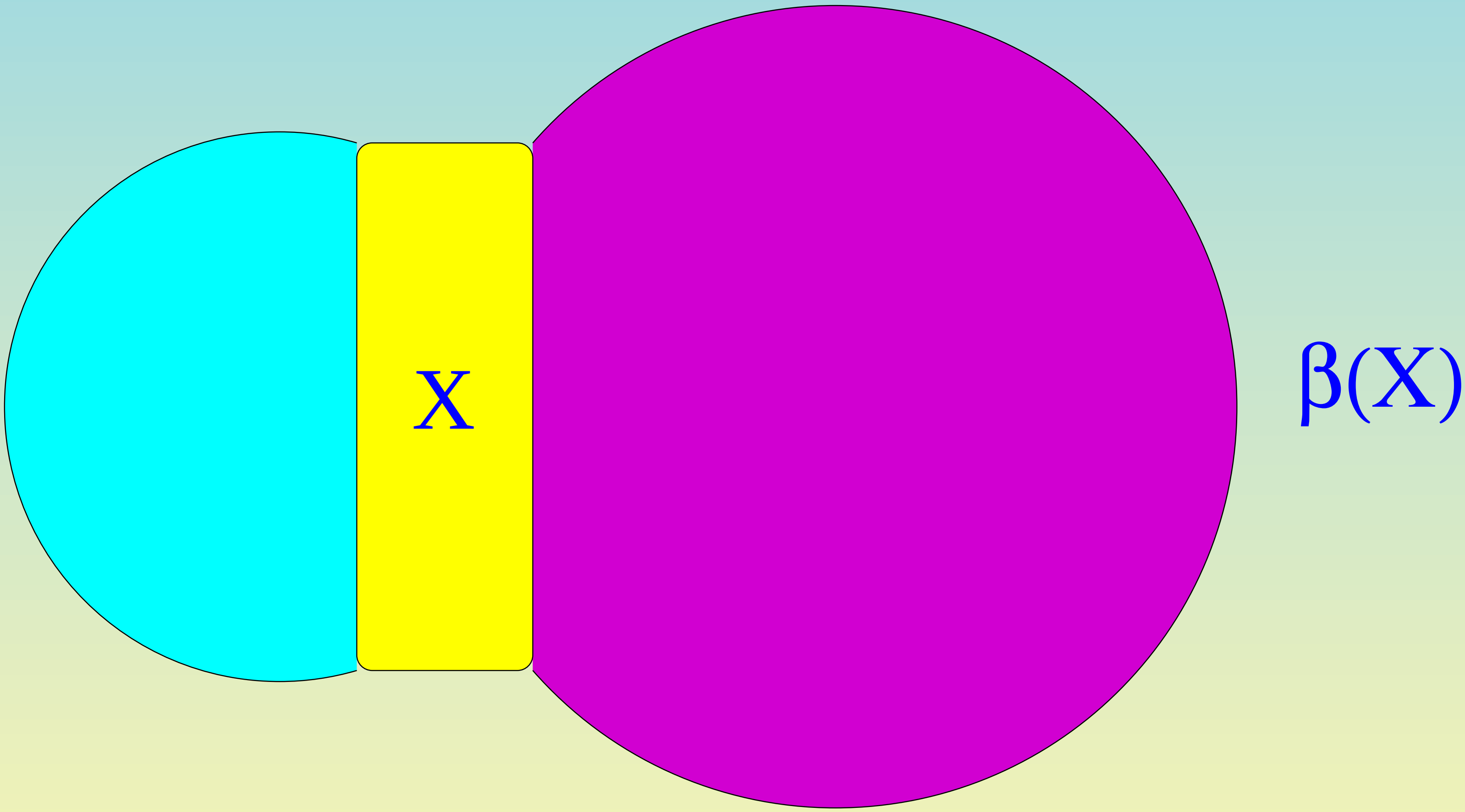

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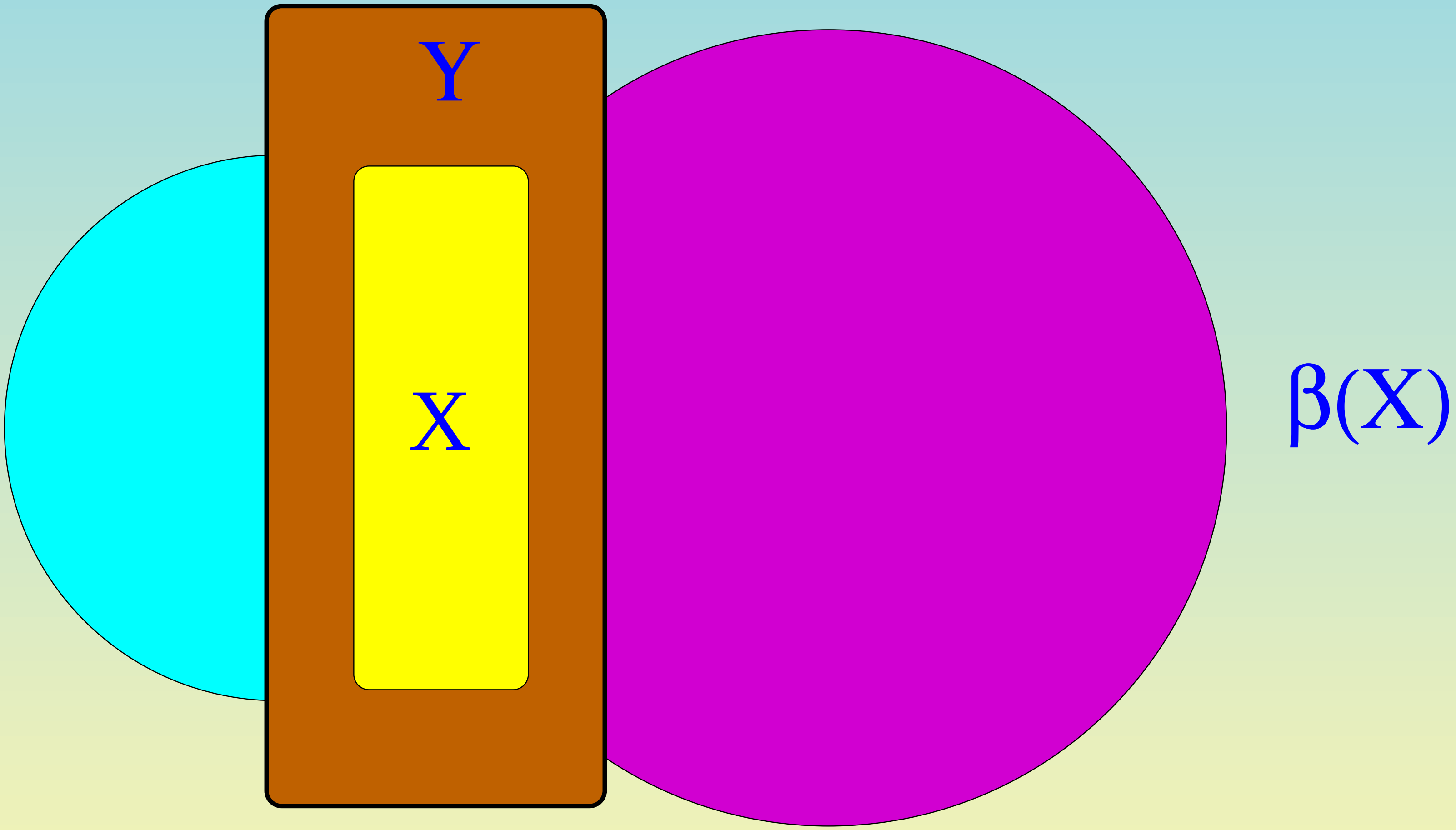

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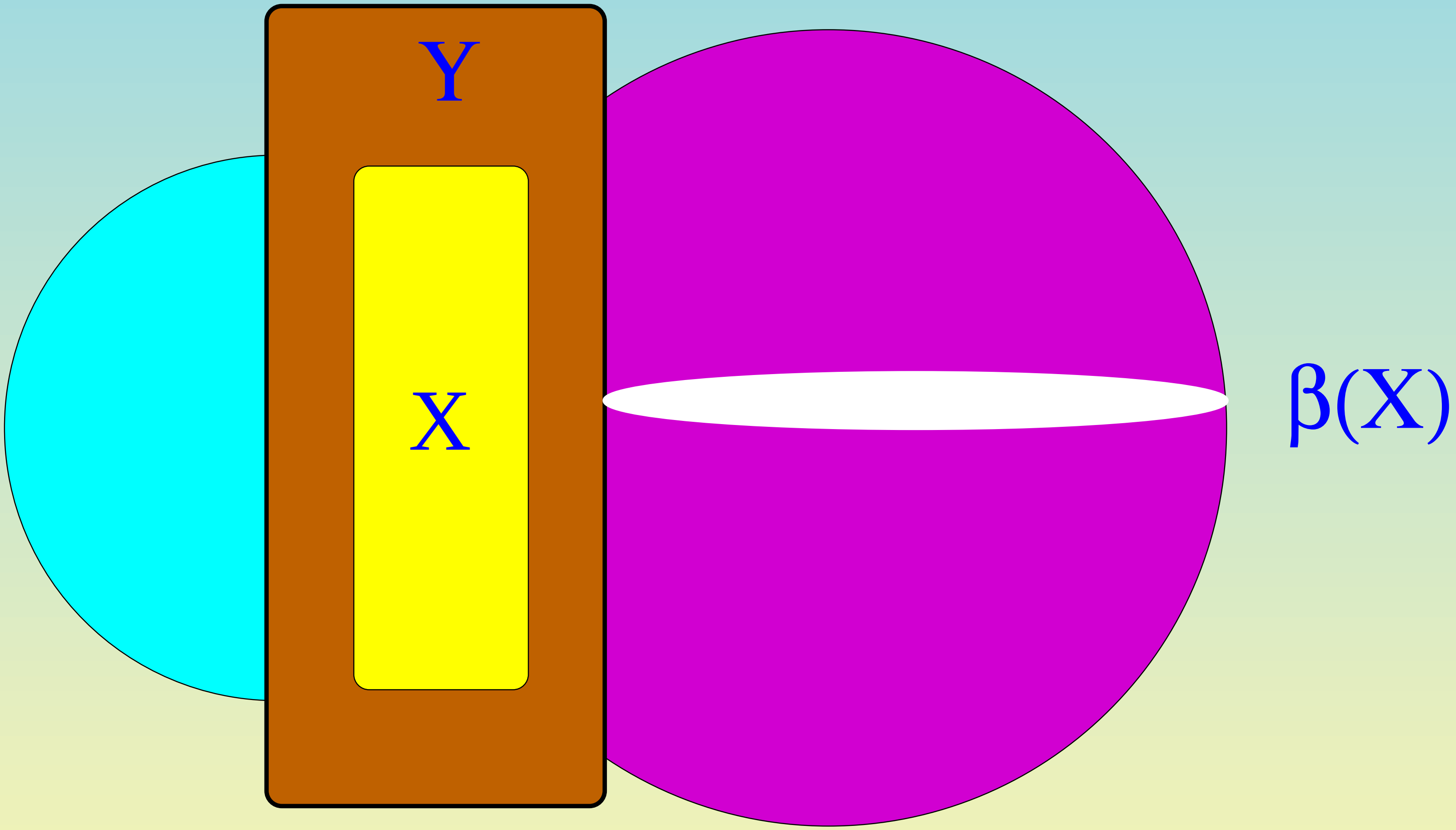

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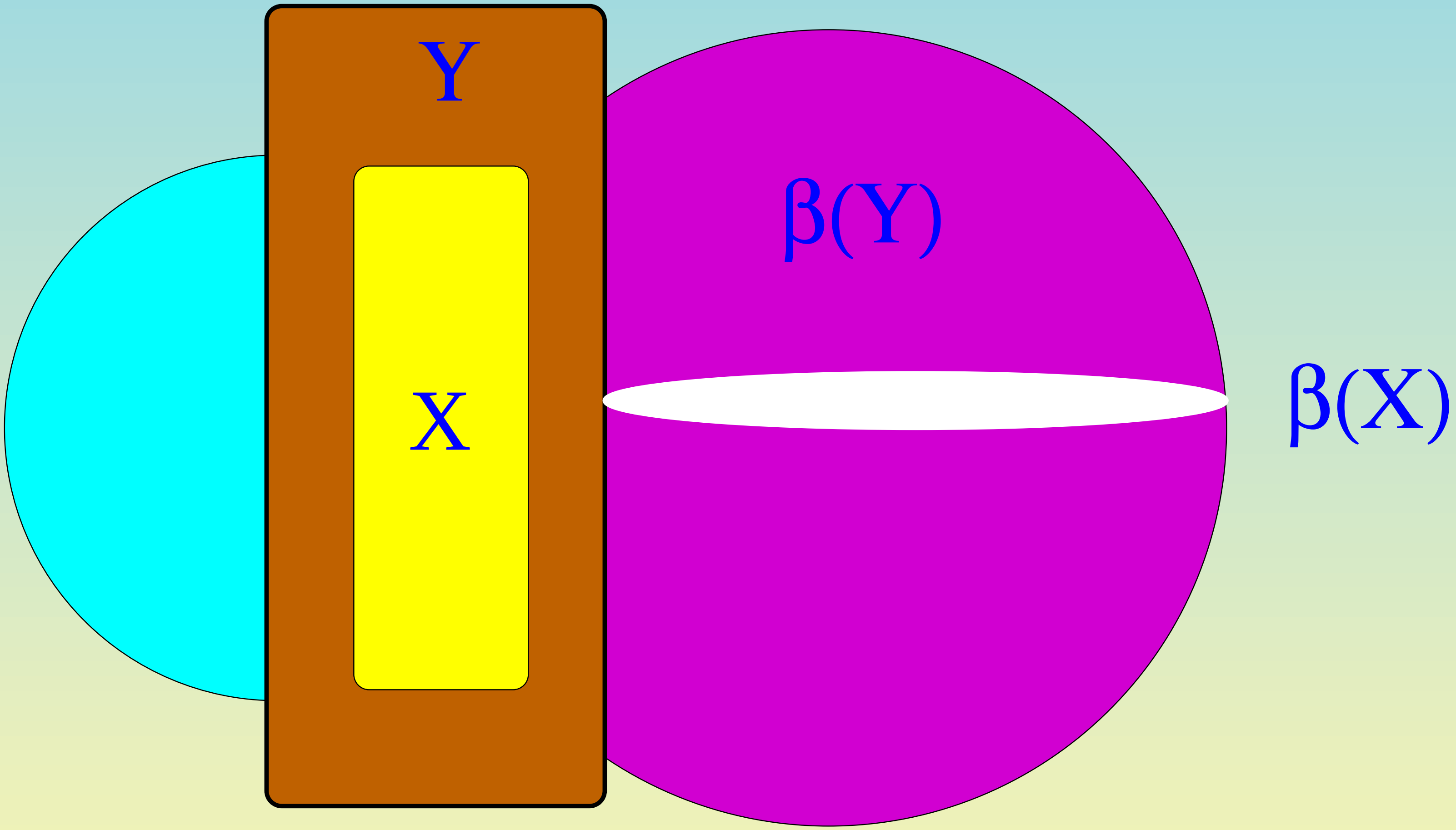

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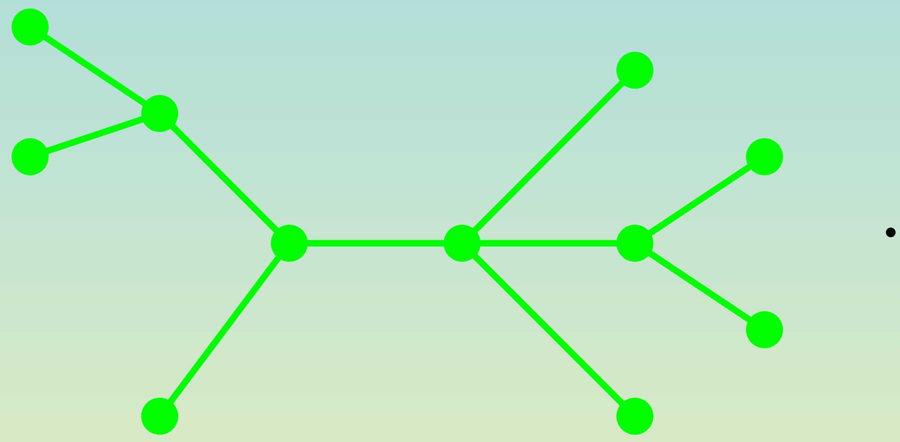

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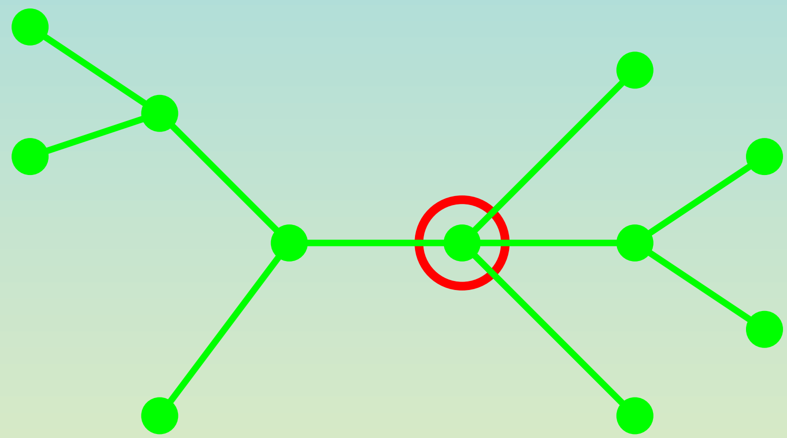

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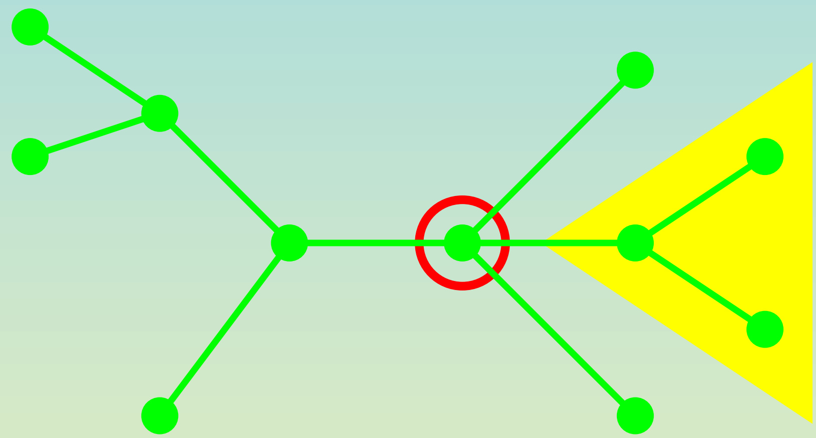

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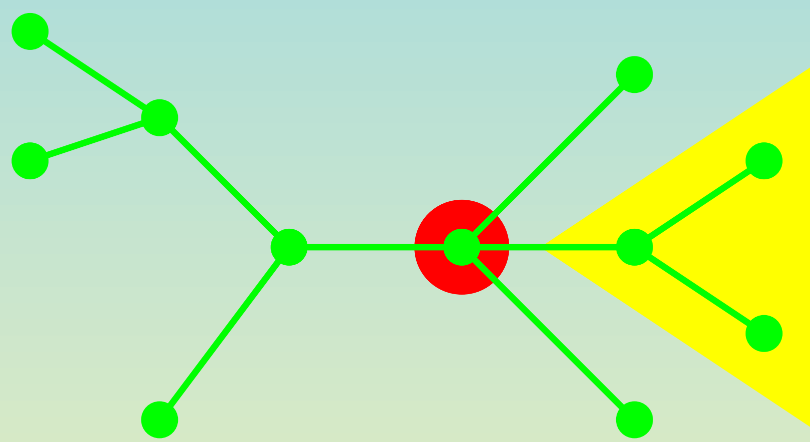

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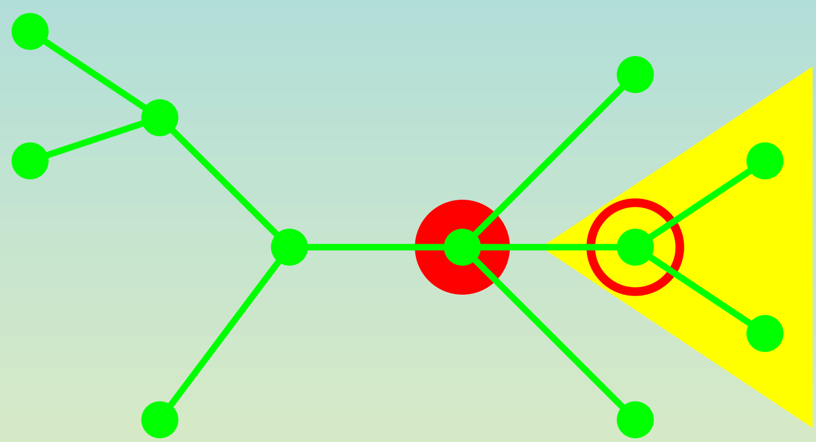

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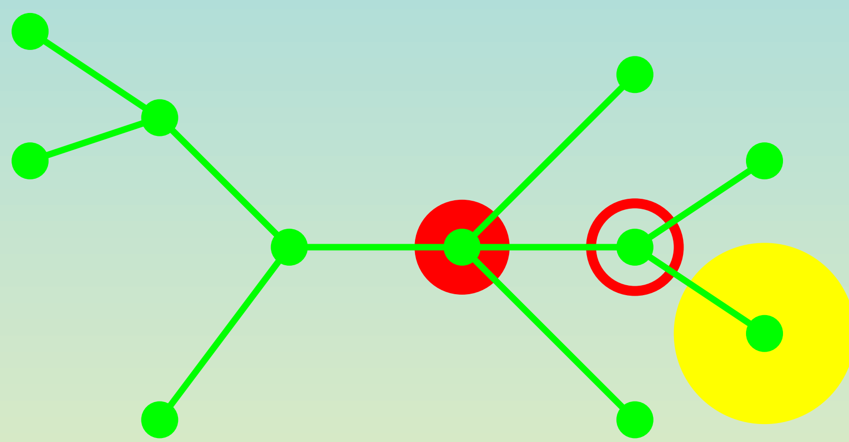

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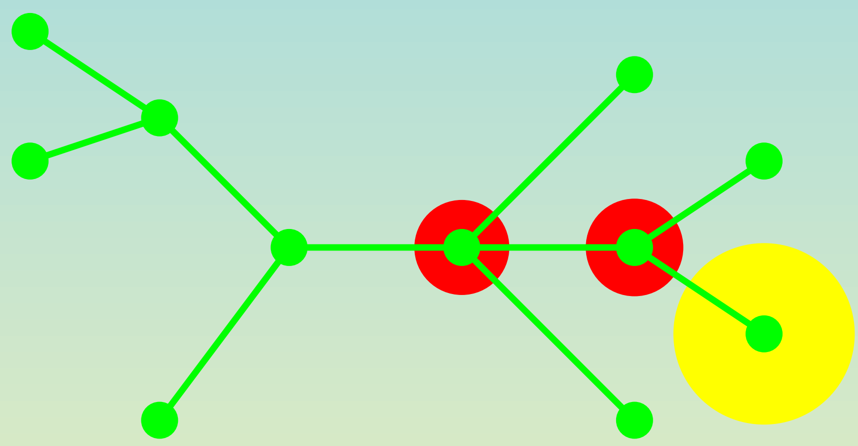

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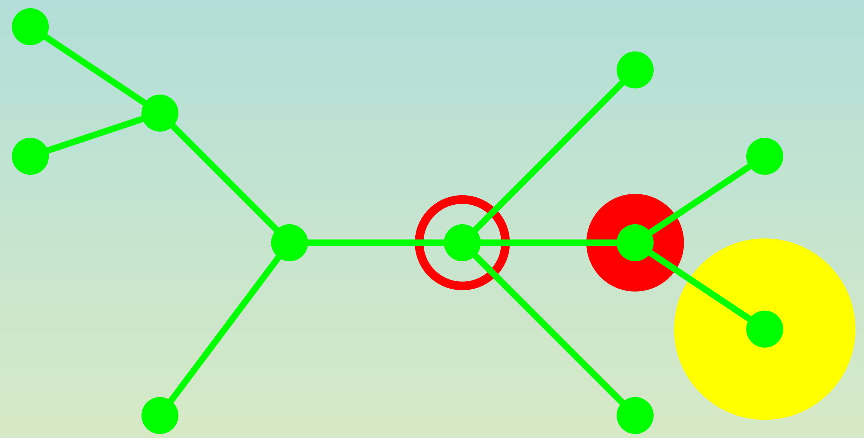

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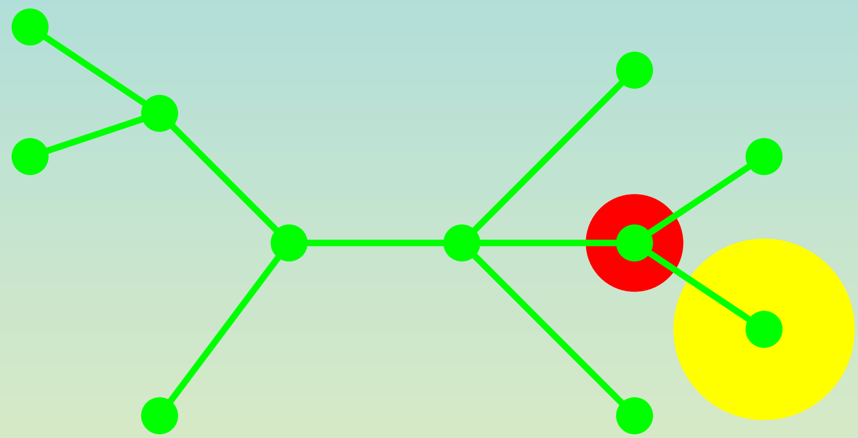

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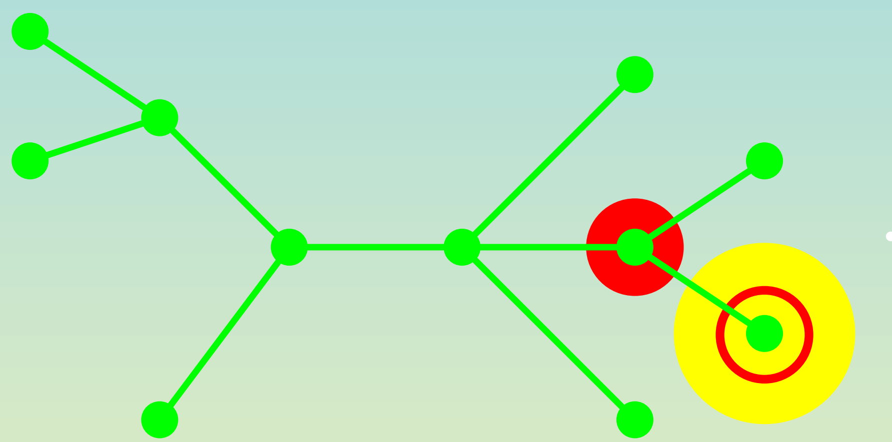

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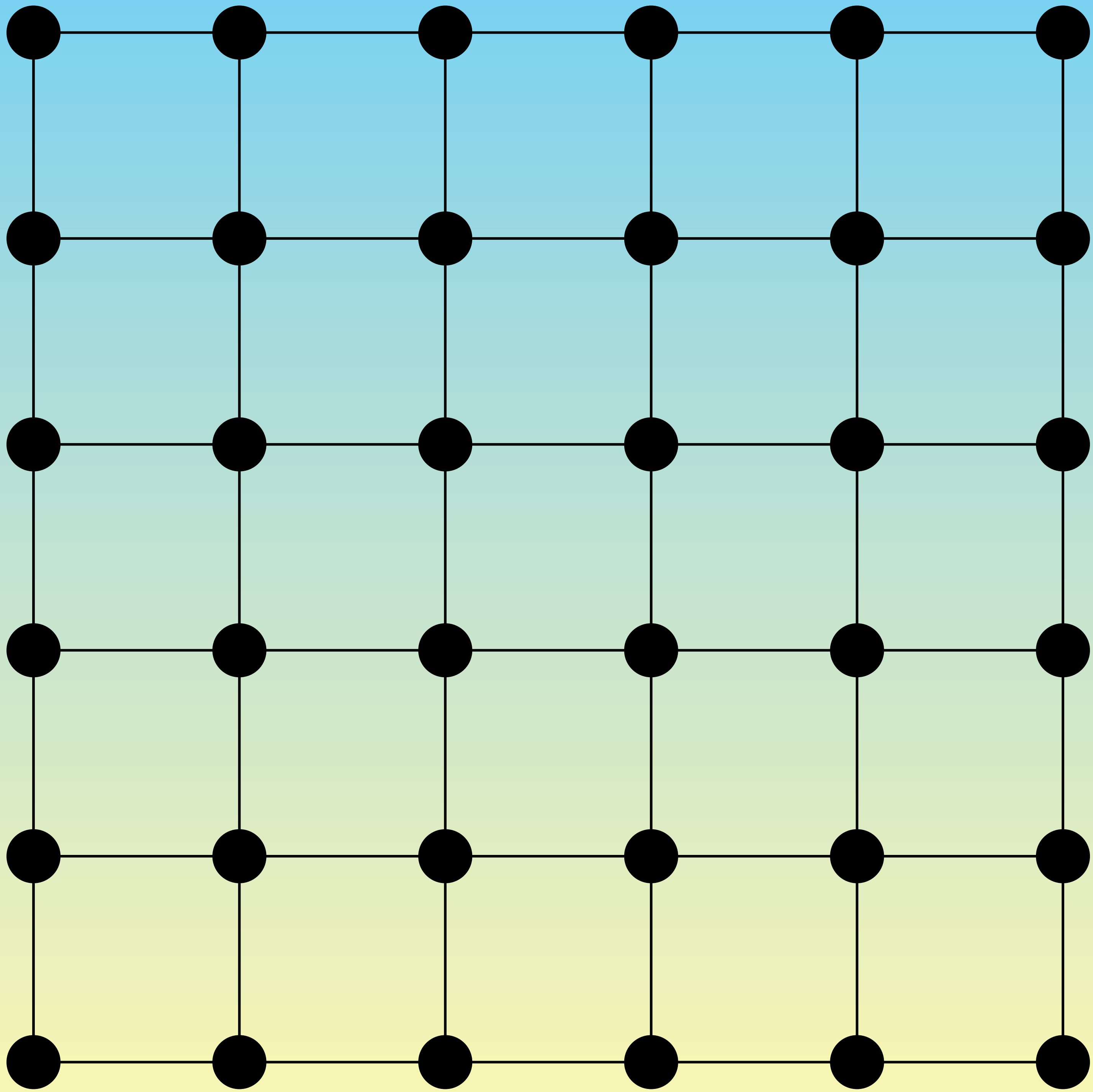
COR Search strategy $\Rightarrow$ monotone search strategy.



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THEOREM (Arnborg, Proskurowski, ...)
Many problems can be solved in linear time when restricted to graphs of bounded tree-width.

Tree-width is useful in

- theory
- design of theoretically fast algorithms
- practical computations

FEEDBACK VERTEX-SET FOR FIXED k

INSTANCE A graph G

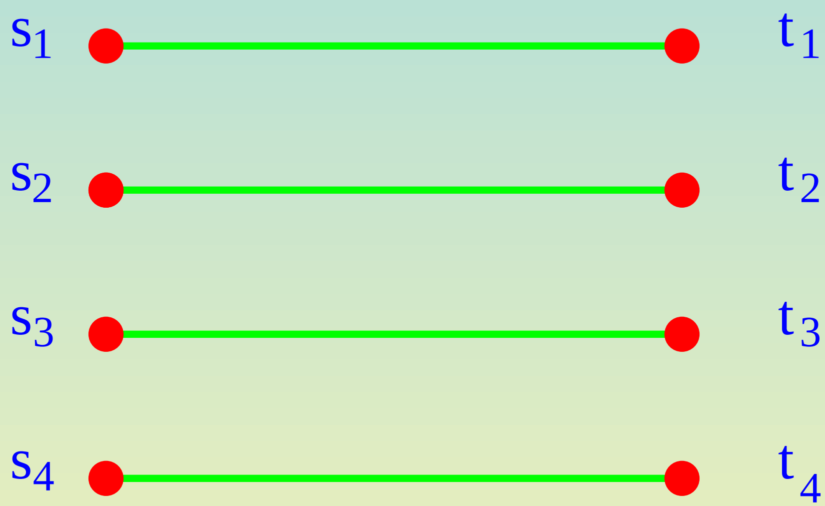
QUESTION Is there a set $X \subseteq V(G)$ such that $|X| \leq k$ and $G \backslash X$ is acyclic?

ALGORITHM If $\text{tw}(G)$ is small use bounded tree-width methods. Otherwise answer "no". That's correct, because big tree-width $\Rightarrow$ big grid $\Rightarrow$ $k+1$ disjoint circuits $\Rightarrow$ X does not exist.

k DISJOINT PATHS IN PLANAR GRAPHS

INSTANCE A planar graph G, vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ of G

QUESTION Are there disjoint paths $P_1, .., P_k$ such that P_i has ends s_i and t_i?



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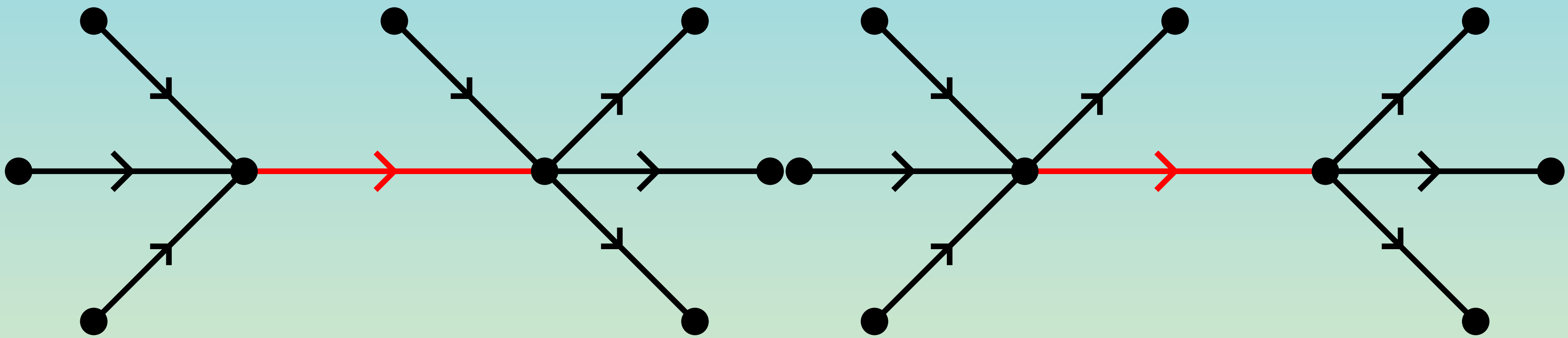
ALGORITHM $\text{tw}(G)$ small $\Rightarrow$ bounded tree-width methods. Otherwise big grid minor $\Rightarrow$ big grid minor with the terminals outside. The middle vertex of this grid minor can be deleted, without affecting the feasibility of the problem.

MINORS IN DIGRAPHS

An edge in a digraph is contractible if either it is the only edge leaving its tail, or it is the only edge entering its head.



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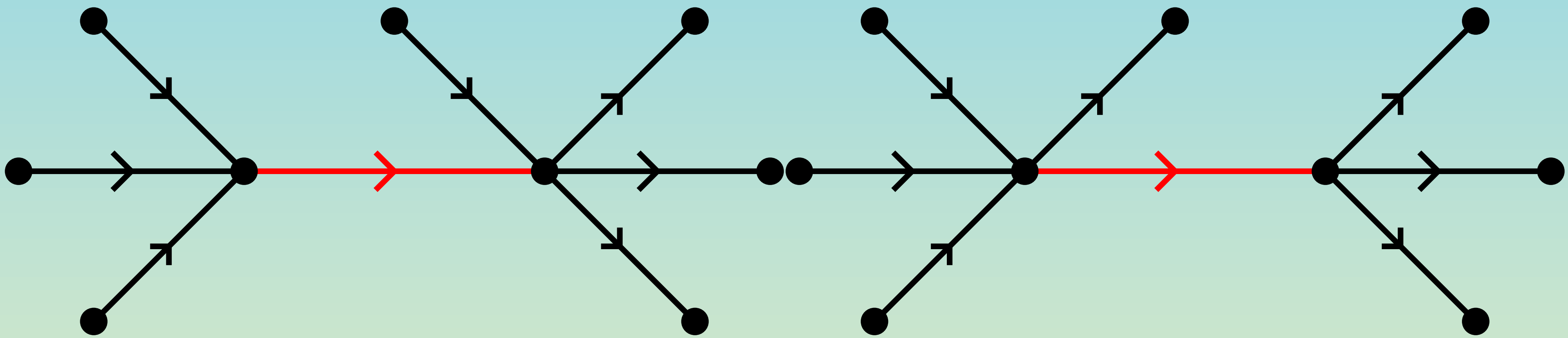

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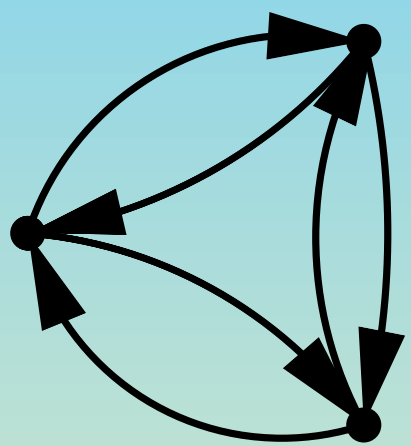
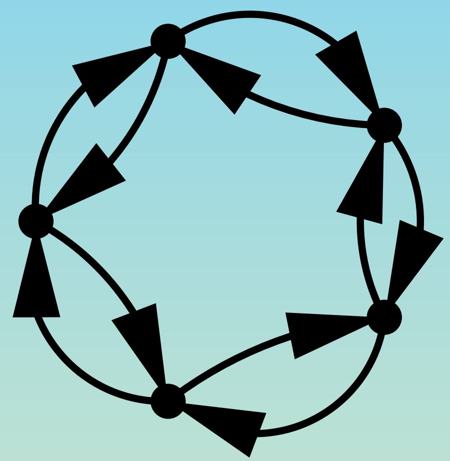
A digraph is even if every subdivision has an even directed circuit. An odd double cycle, O_{2k+1}:





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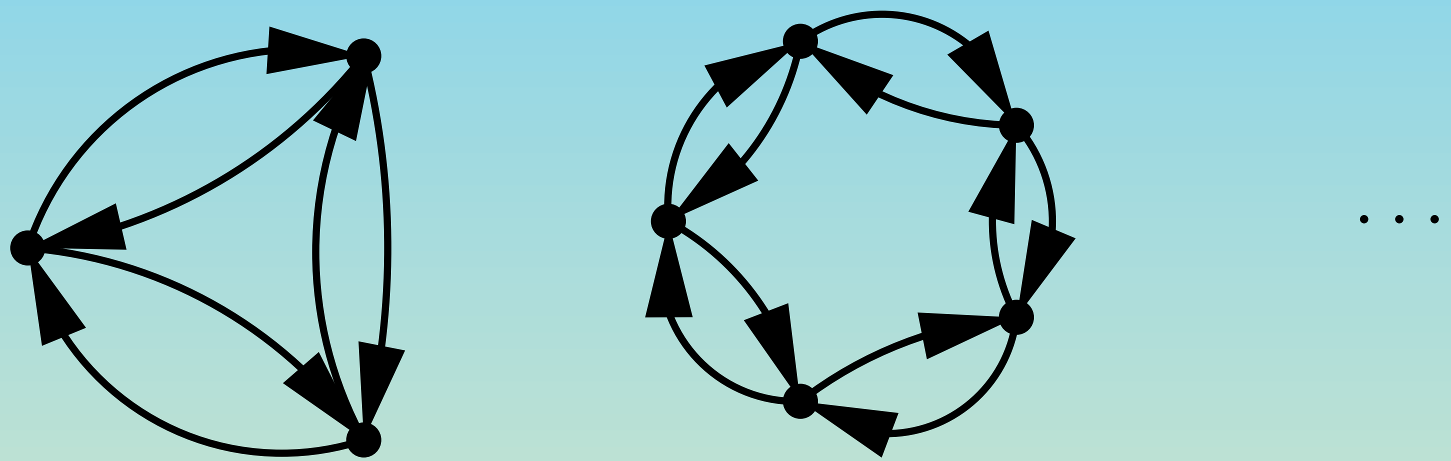

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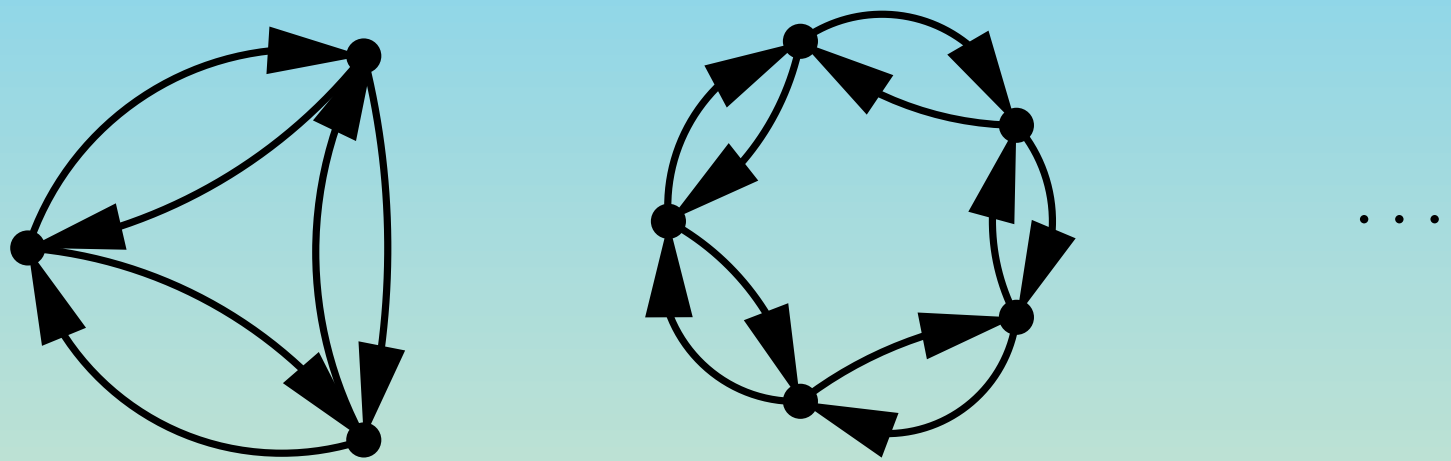

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THEOREM (McCuaig; Robertson, Seymour, RT) $\Leftrightarrow$ it can be obtained from strongly planar digraphs and F_7 by means of 0-, 1-, 2-, 3-, and 4-sums.

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THEOREM (Reed, Robertson, Seymour, RT) There is a function f such that $\tau(D) \leq f(\nu(D))$ for every D.

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THEOREM (Johnson, Robertson, Seymour, RT)
Haven of order $k \Leftarrow \text{tw}(D) \geq 3k-1$.

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REMARK The search strategy need not be monotone.

ALGORITHMS

Let $Z \subseteq V(D)$, and let $S_1, \ldots, S_t$ be the strong components of $D \backslash Z$ such that no edge goes from S_j to S_i for $j > i$. Then $S = S_i \cup S_{i+1} \cup \ldots \cup S_j$ is Z-normal. If $|Z| \leq k$, then S is k-protected.

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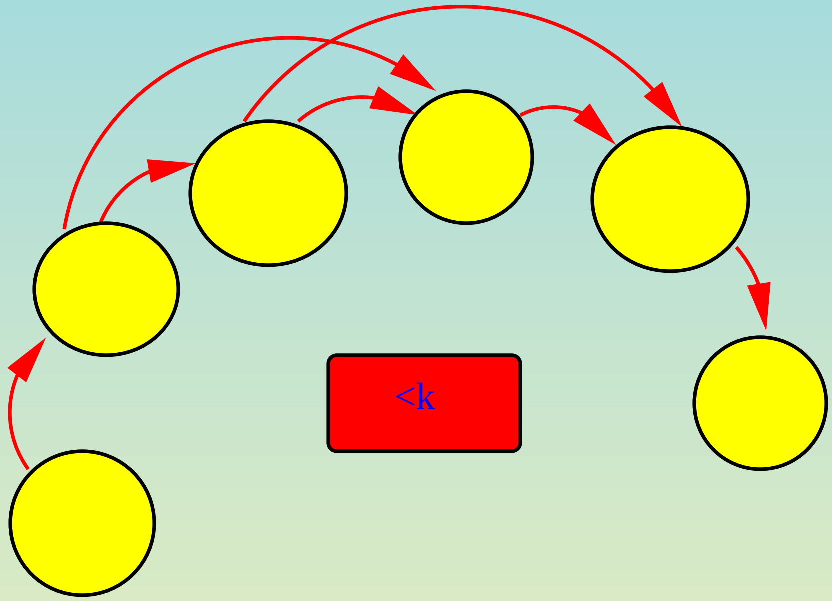

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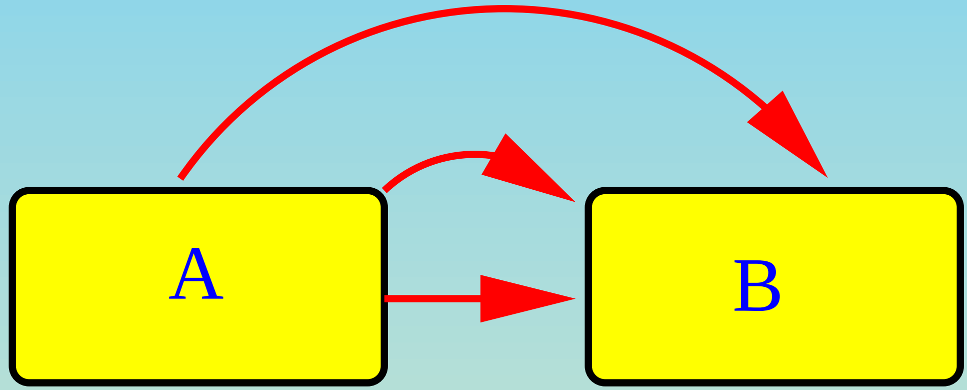
AXIOM 2 $A, B \subseteq V(D)$ disjoint sets, A is k-protected and $|B| \leq k$. Then an itinerary for $A \cup B$ can be computed from itineraries of A and B in time $O((|A| + 1)^{\alpha})$.

AXIOM 1



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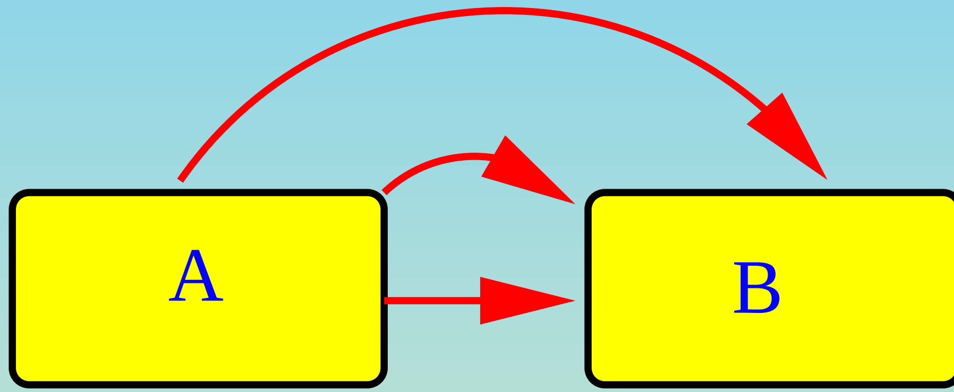

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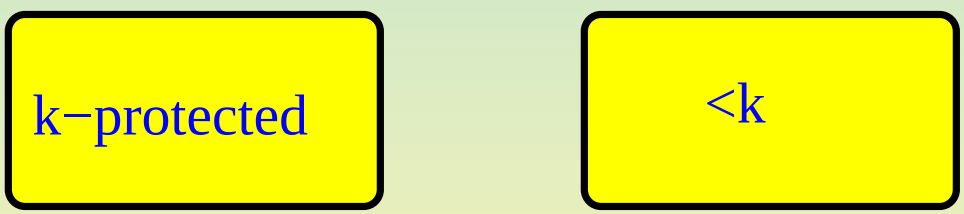

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INPUT A digraph D with an arboreal decomposition of bounded width.

OUTPUT An itinerary for $V(D)$

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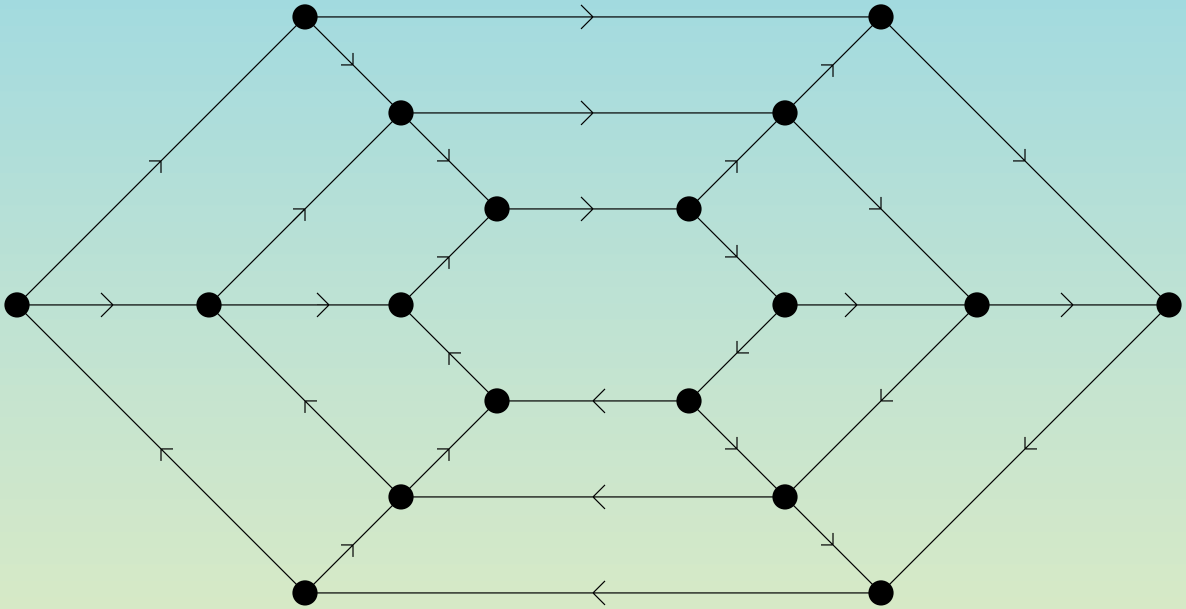
Thus HAMILTON PATH, HAMILTON CIRCUIT, k-DISJOINT PATHS (k fixed) and other problems can be solved in polynomial time for digraphs of bounded tree-width.

CONJECTURE There is a function f such that every digraph of tree-width at least $f(k)$ has a cylindrical $k \times k$ grid minor.

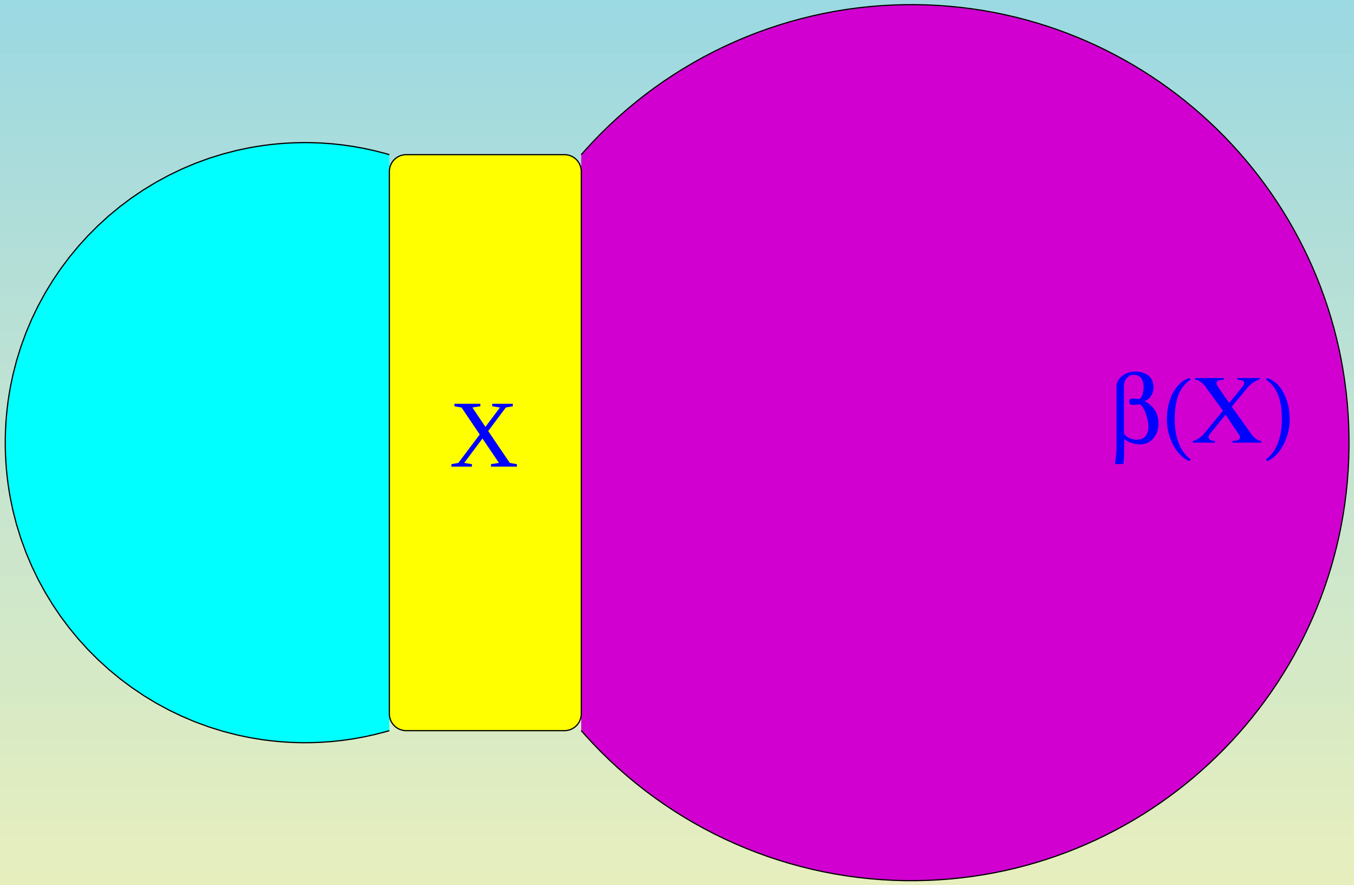


HOW TO USE A HAVEN?

REMINDER A haven β of order k in D assigns to every $X \in [V(D)]^{<k}$ the vertex-set of a strong component of $D \backslash X$ such that

(H) $X \subseteq Y \in [V(D)]^{<k} \Rightarrow \beta(Y) \subseteq \beta(X)$.

Let β be a haven of order k in G. Let $X \subseteq V(G)$ with $|X| \leq k/2$ and $\beta(X)$ minimum. Then X is "externally linked":



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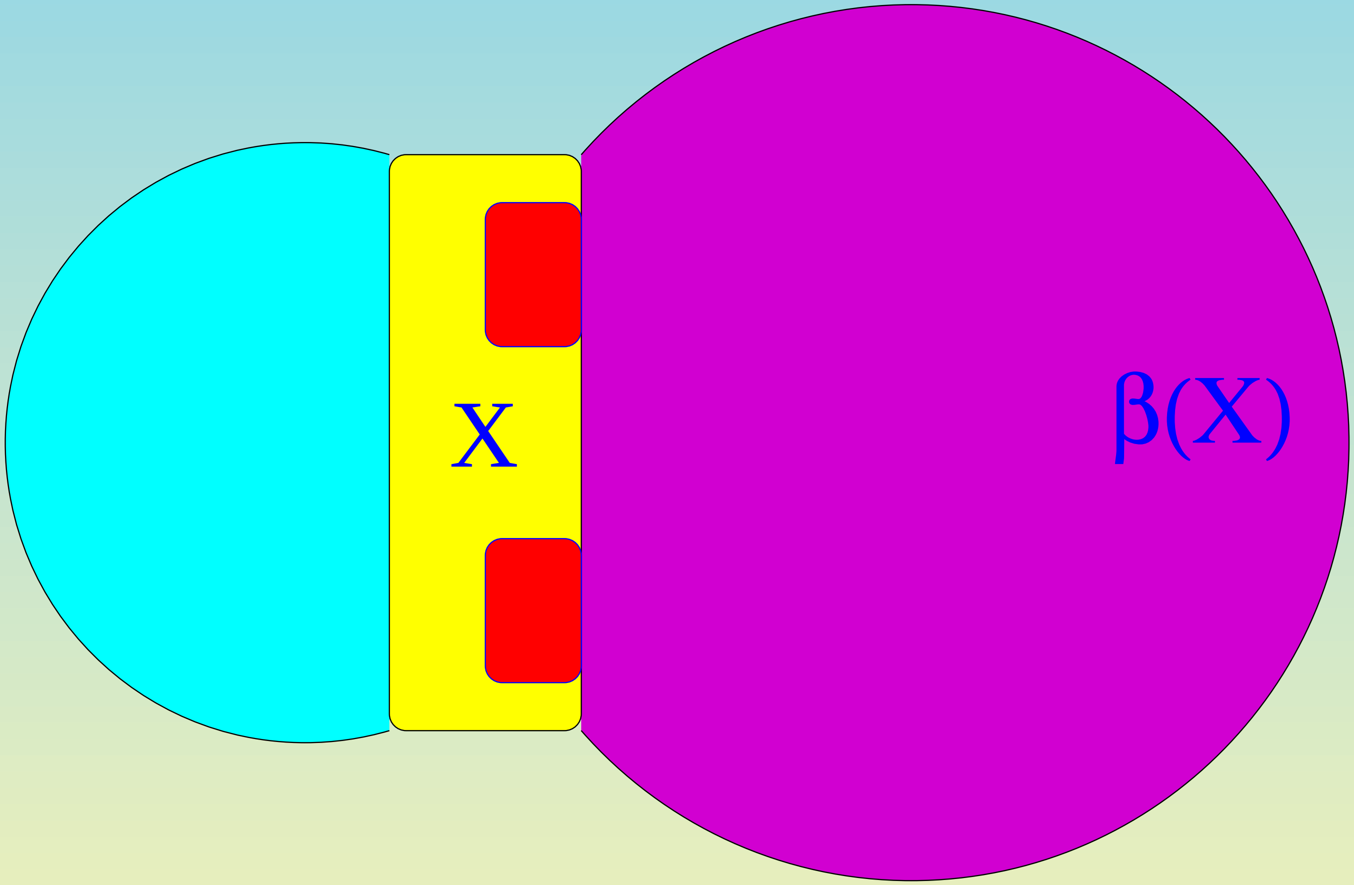

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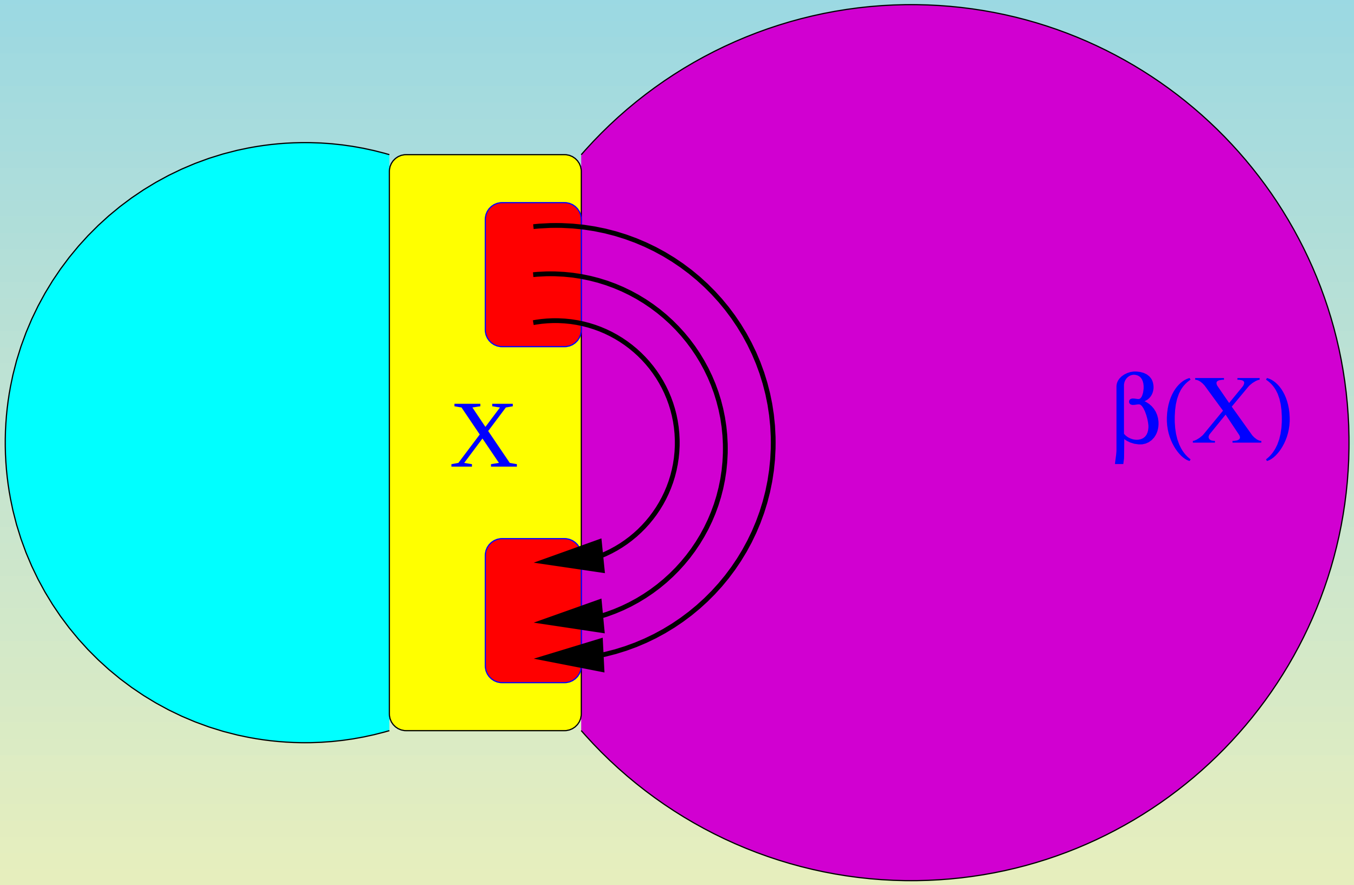

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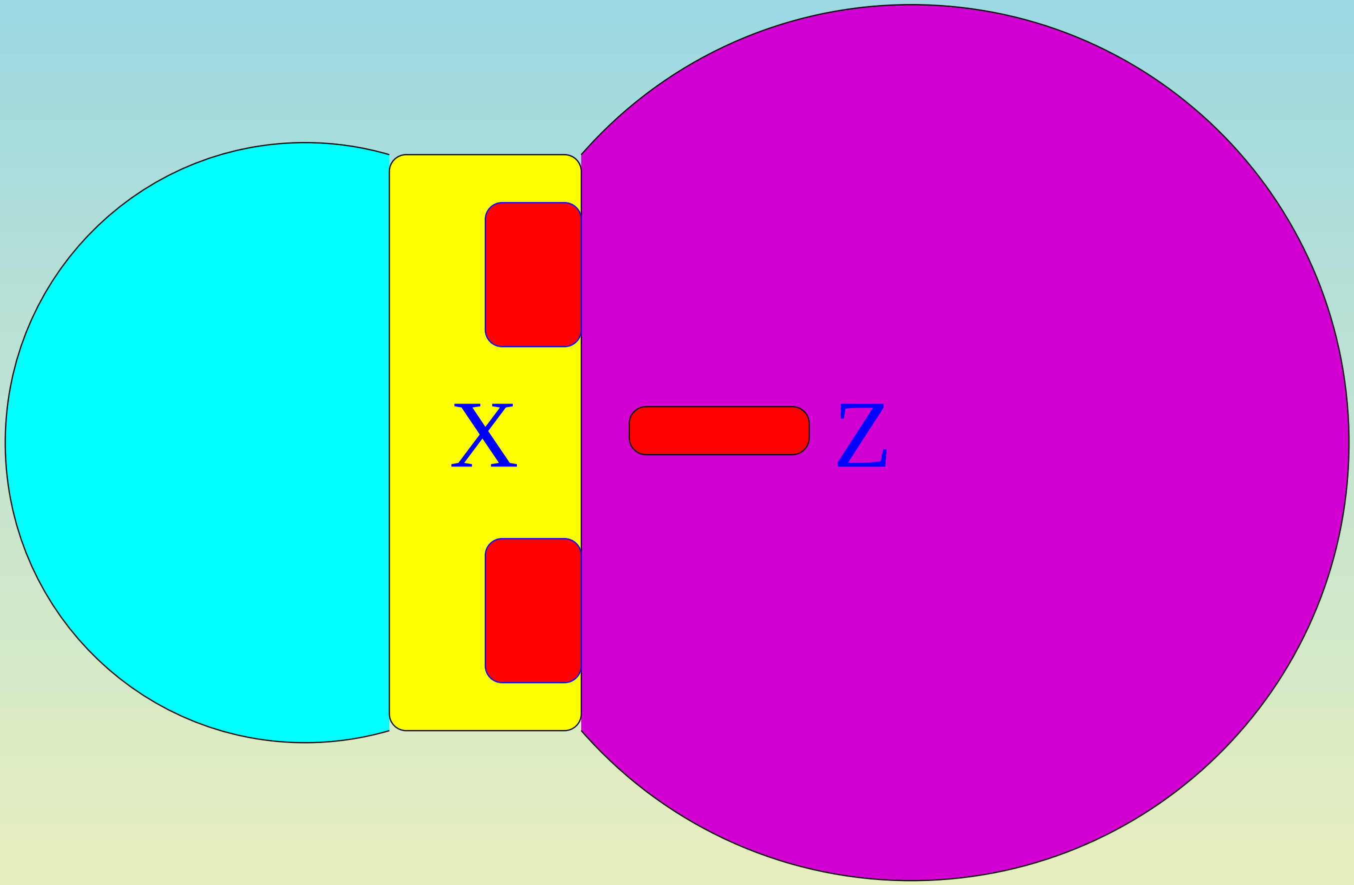

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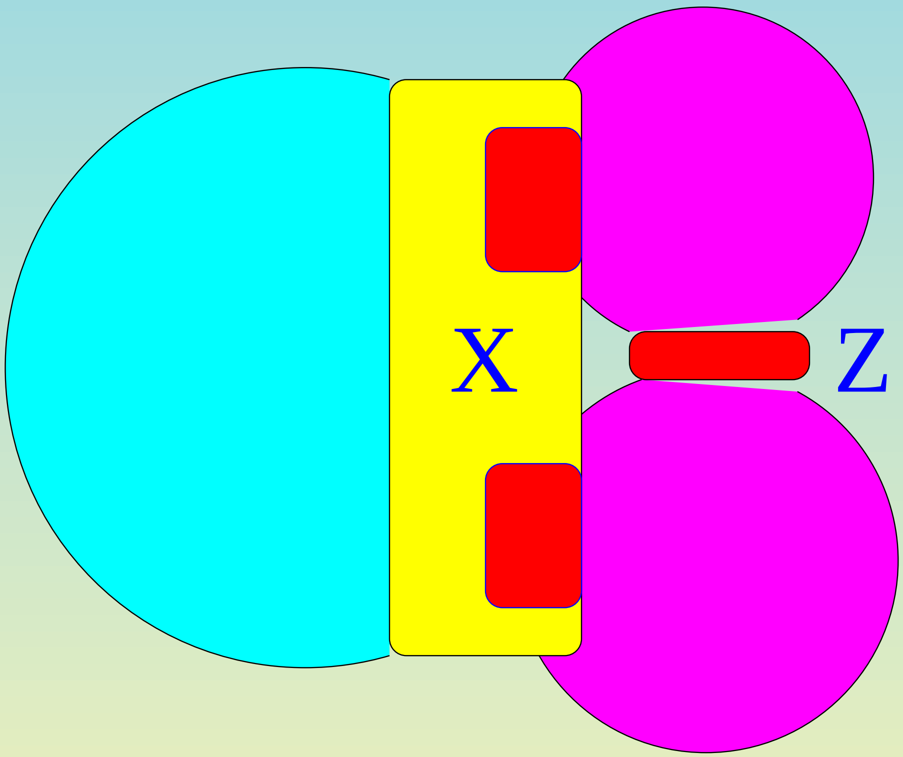

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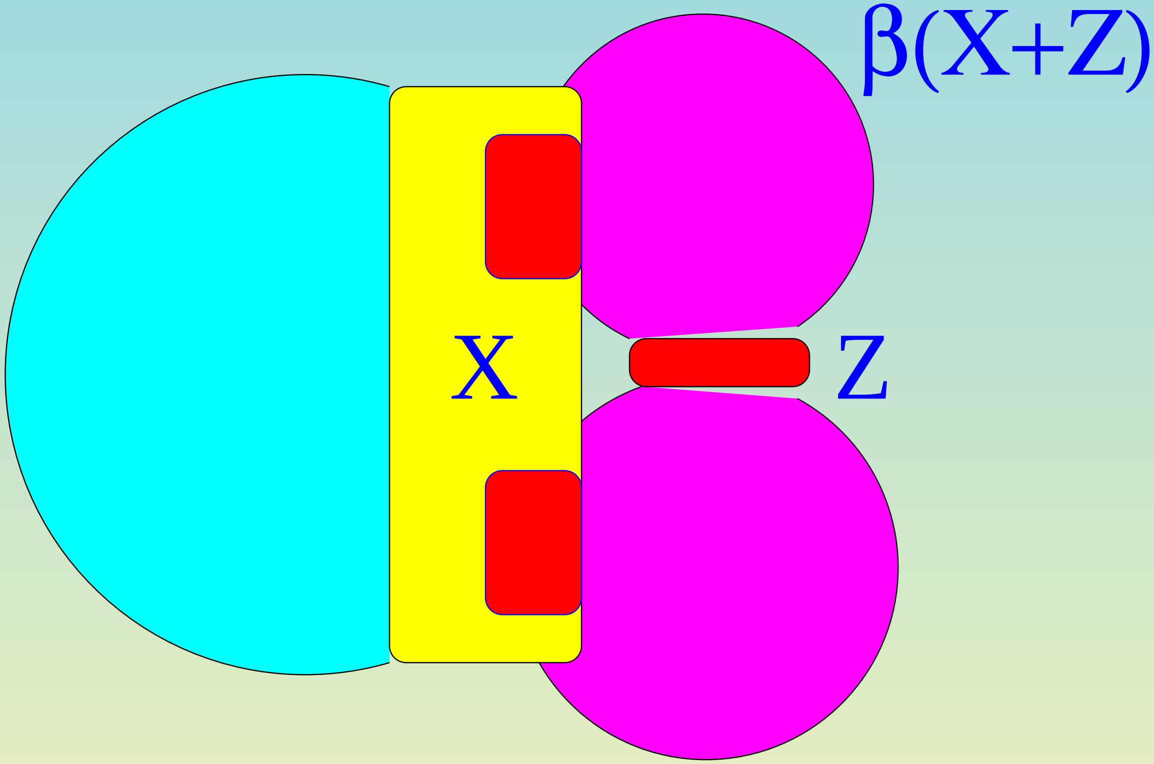

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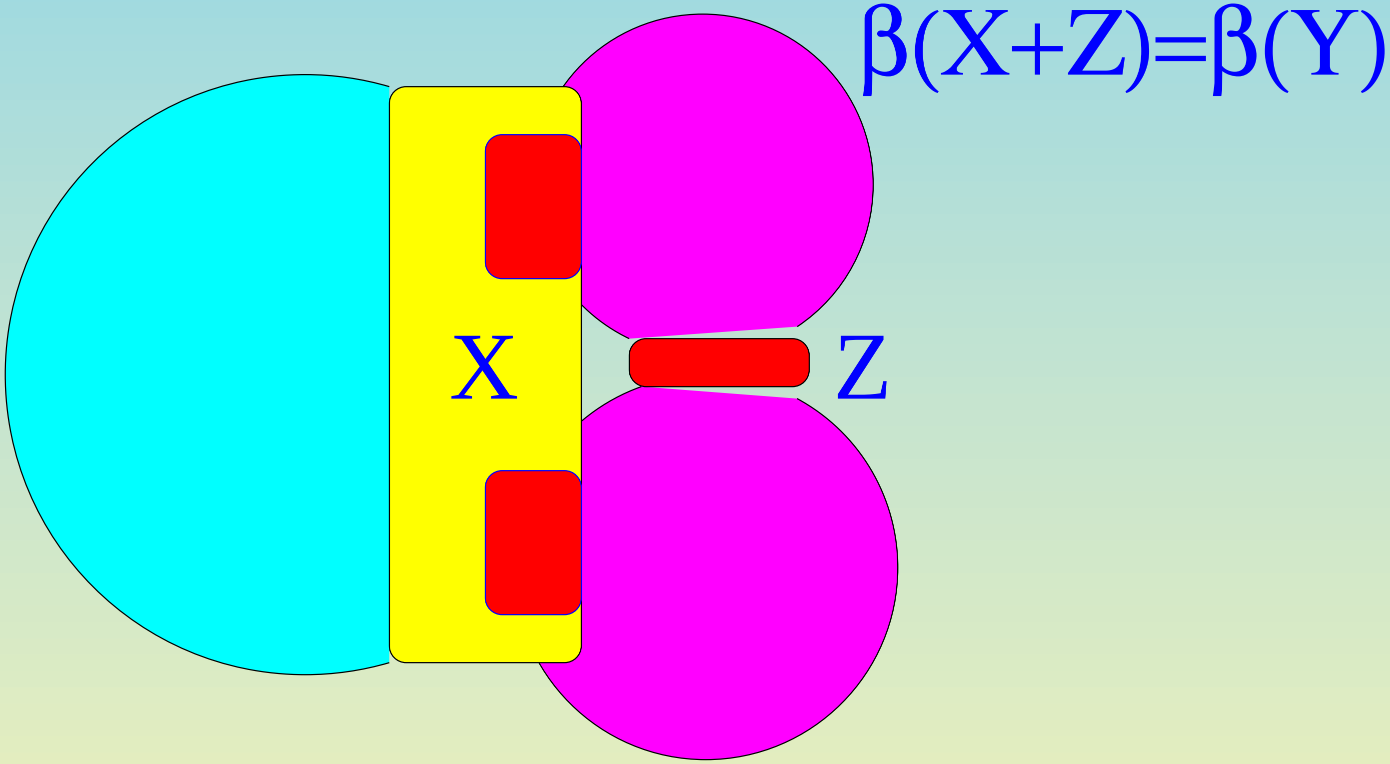

A directed path decomposition of D is a sequence $W_1, W_2, \ldots, W_n$ such that
(i) $\bigcup W_i = V(D)$,
(ii) if $i < i' < i''$ then $W_i \cap W_{i''} \subseteq W_{i'}$,
(iii) for every edge $uv \in E(D)$ there exist $i \leq j$ such that $u \in W_i$ and $v \in W_j$.

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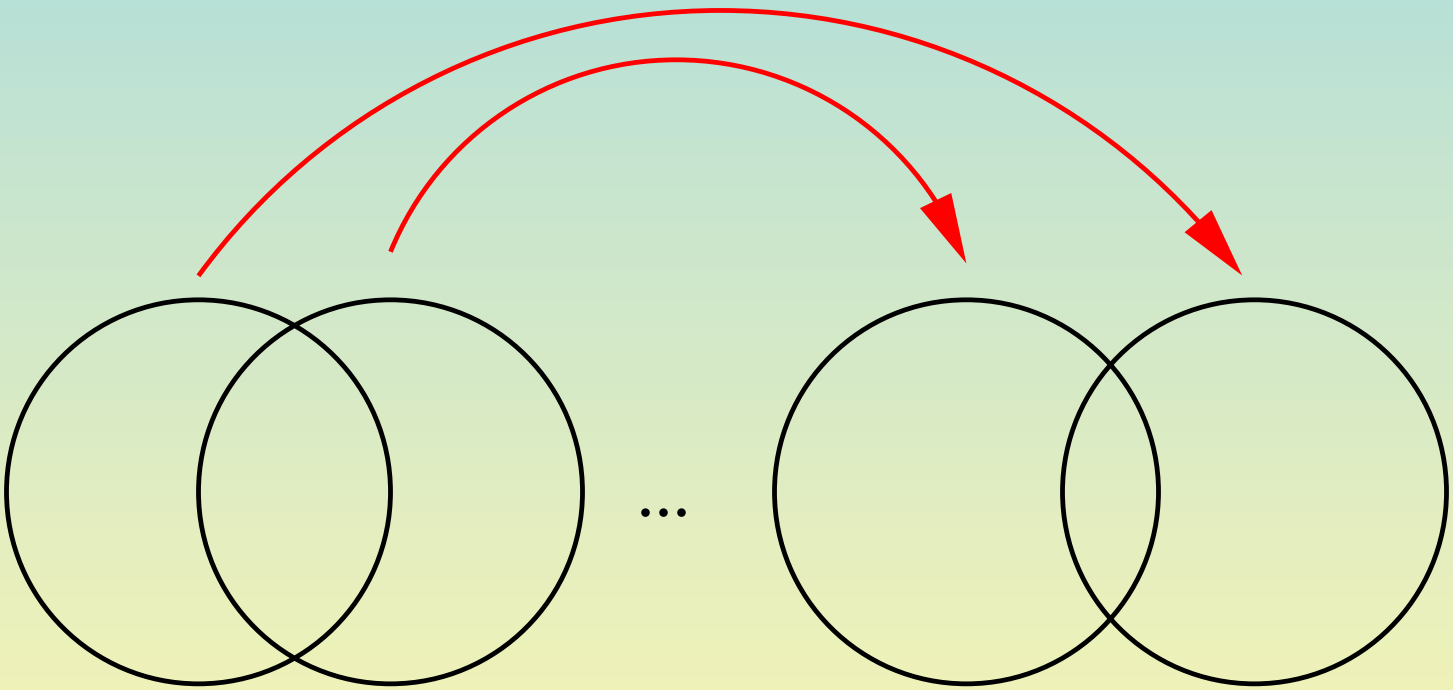

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The directed path-width of D is the minimum width of a directed path-decomposition.

CONJECTURE Big directed path-width $\Rightarrow$ big cylindrical grid minor or a big binary tree minor with each edge replaced by two antiparallel edges.