WELL-QUASI-ORDERING

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GRAPH MINOR THEOREM

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Finite graphs are well-quasi-ordered by $\leq_m$.

A quasi-order is $(Q, \leq)$, where $\leq$ is reflexive and transitive.

NOTE Let $x \equiv y$ mean $x \leq y$ and $y \leq x$. Then $Q/\equiv$ is a partial order. Define $x < y$ to mean $x \leq y$ and $y \not\leq x$.

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$(Q, \leq)$ is well-quasi-ordered (wqo) if for every infinite sequence $q_1, q_2, \ldots$ there exist $i < j$ with $q_i \leq q_j$.

NOTE Equivalent to

- no infinite antichain, and
- no infinite descending sequence $q_1 > q_2 > \cdots$

LEMMA If $(Q, \leq)$ is wqo, then for every infinite sequence $q_1, q_2, \ldots$ there exist $i_1 < i_2 < \cdots$ with $q_{i_1} \leq q_{i_2} \leq \cdots$.

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PROOF Say i is terminal if $q_i \leq q_j$ for no $j > i$. There are only finitely many terminal indices. Let i_1 be larger than all terminal indices. $\exists\ i_2 > i_1$ with $q_{i_1} \leq q_{i_2}$ $\exists\ i_3$ with $q_{i_2} \leq q_{i_3}$, etc.

LEMMA If $(Q_1, \leq_1)$ and $(Q_2, \leq_2)$ are wqo, then $(Q_1 \times Q_2, \leq)$ is wqo. Here $(q_1, q_2) \leq (q_1', q_2')$ if $q_1 \leq_1 q_1'$ and $q_2 \leq_2 q_2'$.

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PROOF Let $(x_1, y_1), (x_2, y_2), \ldots$ be given. Find $i_1 < i_2 < \cdots$ with $x_{i_1} \leq_1 x_{i_2} \leq_1 \cdots$ Find $r < s$ with $y_{i_r} \leq_2 y_{i_s}$. Then

$$(x_{i_r}, y_{i_r}) \leq (x_{i_s}, y_{i_s}).$$

THEOREM (Higman) If Q is wqo, then $Q^{<w}$ is wqo.
$Q^{<w} =$ finite sequences of elements of Q, quasi-ordered by monotone domination:

$$(x_1, x_2, \ldots, x_k) \leq (y_1, y_2, \ldots, y_\ell)$$

if there is a strictly increasing mapping $f : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, \ell\}$ such that $x_i \leq y_{f(i)}$.

EXAMPLE $(1, 5, 7, 3) \leq (2, 3, 4, 6, 7, 1, 3)$

PROOF An infinite sequence $s_1, s_2, \ldots$ of elements of $Q^{\leq w}$ is bad if it violates the definition of wqo. We want to choose a minimal bad sequence.
Let $s_1 \in Q^{<w}$ be shortest such that s_1 starts a bad sequence.
Let $s_2 \in Q^{<w}$ be shortest such that s_1, s_2 starts a bad sequence.
Let $s_3 \in Q^{<w}$ be shortest such that s_1, s_2, s_3 starts a bad sequence.
etc.
Let $s_i = q_i + s_i'$

CLAIM $\{s'_1, s'_2, \ldots\}$ is wqo

PROOF OF CLAIM Let $s'_{i_1}, s'_{i_2}, \ldots$ be a bad sequence. WMA $i_1 < i_2 < \cdots$ Then

$$s_1, s_2, \ldots, s_{i_1 - 1}, s'_{i_1}, s'_{i_2}, \ldots$$

is a bad sequence, contrary to the choice of s_{i_1}.

By the product theorem $Q \times \{s'_1, s'_2, \ldots,\}$ is wqo. So $\exists\ i < j\ q_i \leq q_j$ and $s'_i \leq s'_j$. But then $s_i \leq s_j$, as required.

TOPOLOGICAL CONTAINMENT ON ROOTED TREES

$T_1 \leq_t T_2$ if $\exists$ a 1-1 mapping $f : V(T_1) \rightarrow V(T_2)$ such that $f(t_1 \wedge t_2) = f(t_1) \wedge f(t_2)$.

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PROOF Choose e_1 as high as possible such that it starts a section. Choose e_2 as high as possible such that e_1, e_2 start a section. Etc.

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LEMMA Let $e_1, e_2, \ldots$ be a minimal bad section, and let e'_i, e''_i be the daughters of e_i. Then $\{\uparrow e'_1, \uparrow e'_2, \ldots\}$ and $\{\uparrow e''_1, \uparrow e''_2, \ldots\}$ are wqo.

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PROOF WMA there is a bad section $e'_{i_1}, e'_{i_2}, \ldots$. Then $e_1, e_2, \ldots, e_{i_1-1}, e'_{i_1}, e'_{i_2} \ldots$ contradicts minimality.

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PROOF OF KRUSKAL'S THM By the product theorem there exist $i < j$ such that $\uparrow e'_i \leq_t \uparrow e'_j$, and $\uparrow e''_i \leq_t \uparrow e''_j$. But then $\uparrow e_i \leq_t \uparrow e_j$ contrary to badness.

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PROOF Same as above; minimality somewhat trickier.

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COR Planar graphs are wqo by $\leq_m$.

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FACT Not known even when every component is finite.

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THM Friedman Above unprovable in Peano arithmetic.