COLORING TRIANGLE-FREE GRAPHS ON SURFACES

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joint work with
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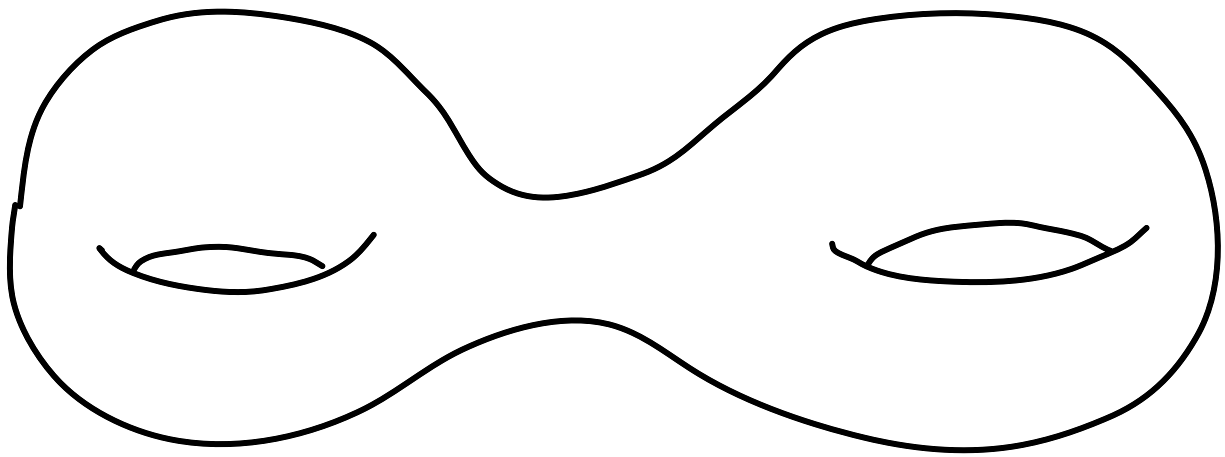
A surface is a compact 2-manifold. $G \hookrightarrow \Sigma$ means *G* is drawn in Σ with no crossing.

The classification theorem.

Every orientable surface is homeomorphic to a sphere with *g* handles. Notation: S_g

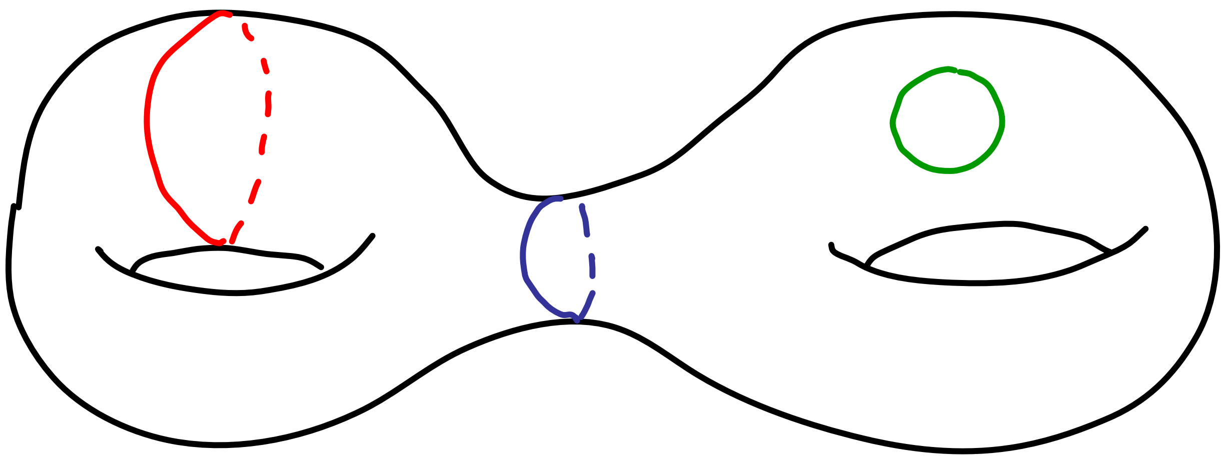
Every nonorientable surface is homeomorphic to a sphere with *k* cross-caps.



Cycles in embedded graphs

Trivial if null-homotopic (bounds a disk)

Otherwise non-trivial. May or may not separate the surface. May be 1-sided or 2-sided



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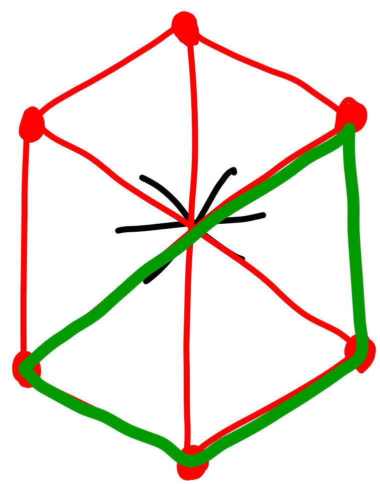

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Coloring graphs on surfaces

The classical problems:

The 4-color problem (Guthrie 1852)

Can every planar graph be 4-colored?

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Two questions for fixed Σ:

(1) Finitely many $(k+1)$-critical $G \hookrightarrow \Sigma$?

(2) Test $\chi(G) \leq k$ in polytime for $G \hookrightarrow \Sigma$?

Yes to (1) implies yes to (2).

Since finitely many $G \hookrightarrow \Sigma$ have min deg >6, yes to (1) for $k \geq 7$. Also yes for $k=6$.

THM (Thomassen) For every Σ, there are only finitely many 6-critical graphs in Σ.

COR $\chi(G) \leq 5$ can be tested in polytime

THM (Fisk) Infinitely many 3-, 4-, 5-critical graphs on any nonplanar surface

THM Testing $\chi(G) \leq 3$ is NP-hard even for planar graphs

Testing $\chi(G) \leq 4$ is open on non-planar surfaces. Prospects not bright.

Triangle-free graphs on surfaces

FACT: For $k \geq 4$ finitely many $(k+1)$-critical triangle-free graphs on any surface.

That leaves the case $k=3$. Now solved.

Triangle-free graphs in the plane

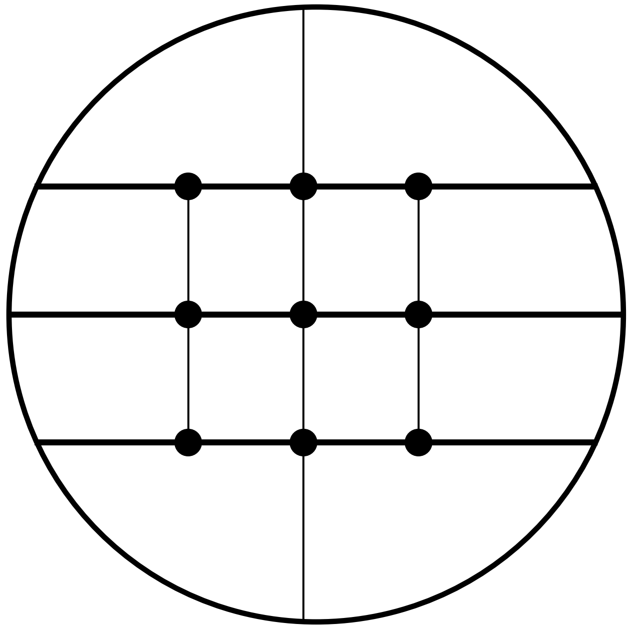
THM (Grotzsch) Every triangle-free planar graph is 3-colorable

The essential step is to prove it for graphs of girth ≥ 5. Short proofs by Thomassen.

Grotzsch's theorem extends to no other surface (Mycielski's construction)

Triangle-free graphs in the projective plane

THM (Youngs) For every non-bipartite quadrangulation G of the projective plane $\chi(G)=4$.



THM (Gimbel, Thomassen) A triangle-free graph in the projective plane is 3-colorable $\Leftrightarrow$ it has no subgraph isomorphic to a non-bipartite quadrangulation of the projective plane.

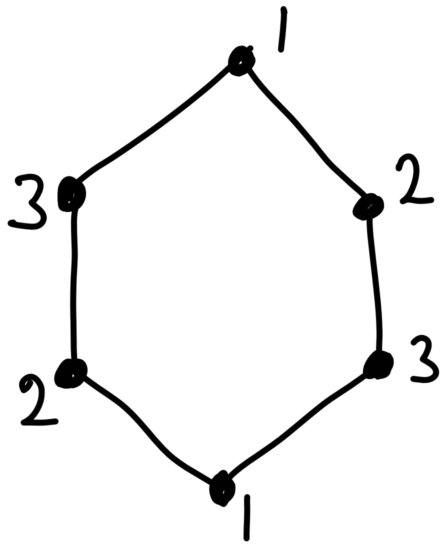
On other surfaces 3-coloring triangle-free graphs was open. The special case of girth at least 5 settled by Thomassen:

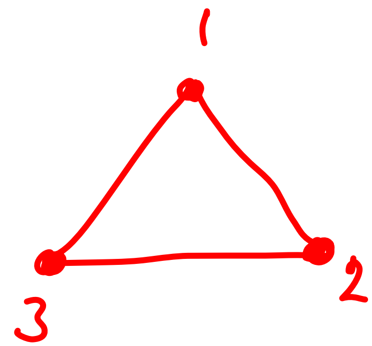
THM (Thomassen) On any Σ there are only finitely many 4-critical graphs of girth ≥ 5

THM (Thomassen) None on the torus.

THM (Walls, RT) None on the Klein bottle.

Let $G \hookrightarrow \Sigma$, Σ orientable, let $c:V(G) \to \{1,2,3\}$ be a 3-coloring. Orient edges $1 \to 2$, $2 \to 3$, $3 \to 1$. Let C be a cycle. Choose clockwise direction of C. Define winding number of C: $w(C) :=$ (#forward edges - #back edges)/3.



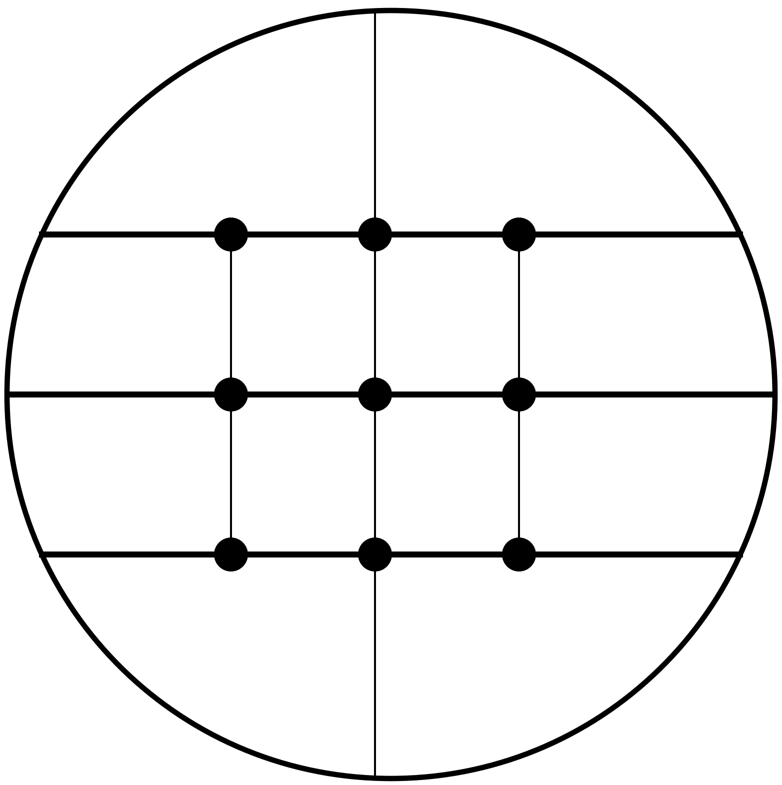


FACT $\sum w(C)=0$, sum over all facial cycles

COR If some faces are precolored, and all other faces are quadrangles, then a necessary condition for the precoloring to extend is that the sum of winding #s be 0.

THM (Youngs) For every non-bipartite quadrangulation G of the projective plane $\chi(G)=4$.

Proof. Cut along a non-trivial cycle C. Obtain a bipartite planar near-quadrangulation with outer cycle C' of length $2|V(C)|$. Since C is odd, the winding number of C' is twice an odd integer. Every other cycle has winding number 0.



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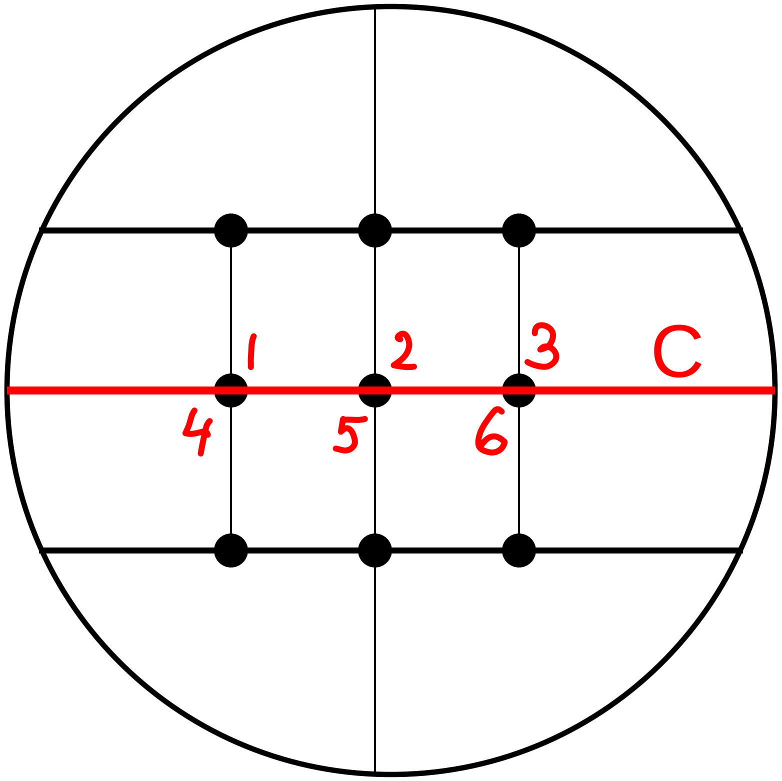

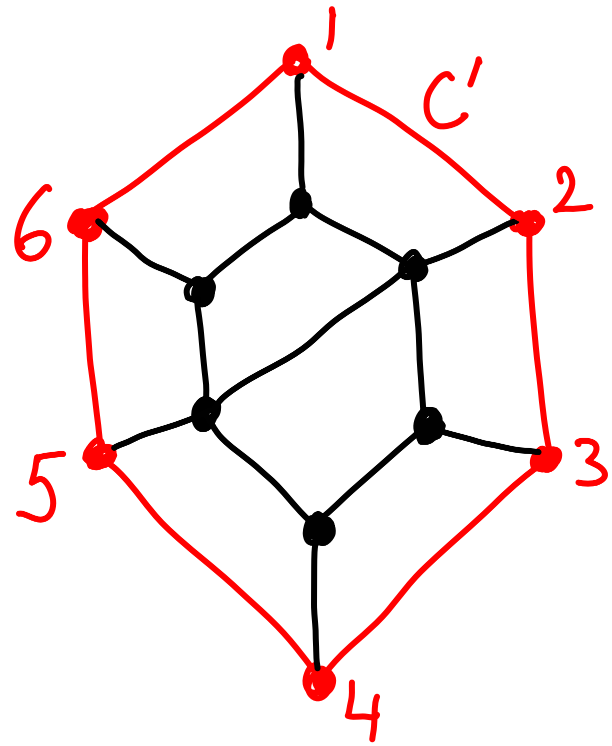

$G \hookrightarrow \Sigma$ is embedded with edge-width $\geq w$ if every non-trivial cycle has length $\geq w$.

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THM 1 (Kral, RT) $\forall \Sigma$ orientable $\forall k\ \exists r\ \forall G \hookrightarrow \Sigma$ with precolored cycles $C_1,\ldots,C_k$ ("holes") if

1. each hole has size $\leq k$
2. all other faces are C_4s
3. there is no "schism" of length $\leq r$
4. no C_i surrounded by cycle of length $<|C_i|$
5. sum of winding numbers is 0

$\Rightarrow$ precoloring extends to a 3-coloring of G

THM 2 (Dvorak, RT) If $G \hookrightarrow \Sigma$ is 4-critical and has no trivial triangles, then it has $\leq f(\Sigma)$ faces of length >4, and each has length $\leq f(\Sigma)$.

THM 3 $\forall\ \Sigma\ \exists\ N$ s.t. every $G \hookrightarrow \Sigma$ with no trivial triangles is either 3-colorable, or has a subgraph H of size $\leq N$ such that for every face f of H it is easy to decide if a precoloring of bd(f) extends to f.

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Easy to decide means: f is a disk, a cylinder, or satisfies (1)-(4) of Thm 1.

COR $\forall\ \Sigma$ 3-colorability of triangle-free graphs $G \hookrightarrow \Sigma$ can be tested in poly-time

COR $\forall\ \Sigma\ \forall$ triangle-free $G \hookrightarrow \Sigma$ all but $f(\Sigma)$ vertices of G can be 3-colored.

Open problem

Fix a proper minor-closed class F. Can $\chi(G)$ be approximated to within an additive error of 10 in poly-time for $G \in F$?