K_t MINORS IN LARGE t-CONNECTED GRAPHS

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joint work with Sergey Norin

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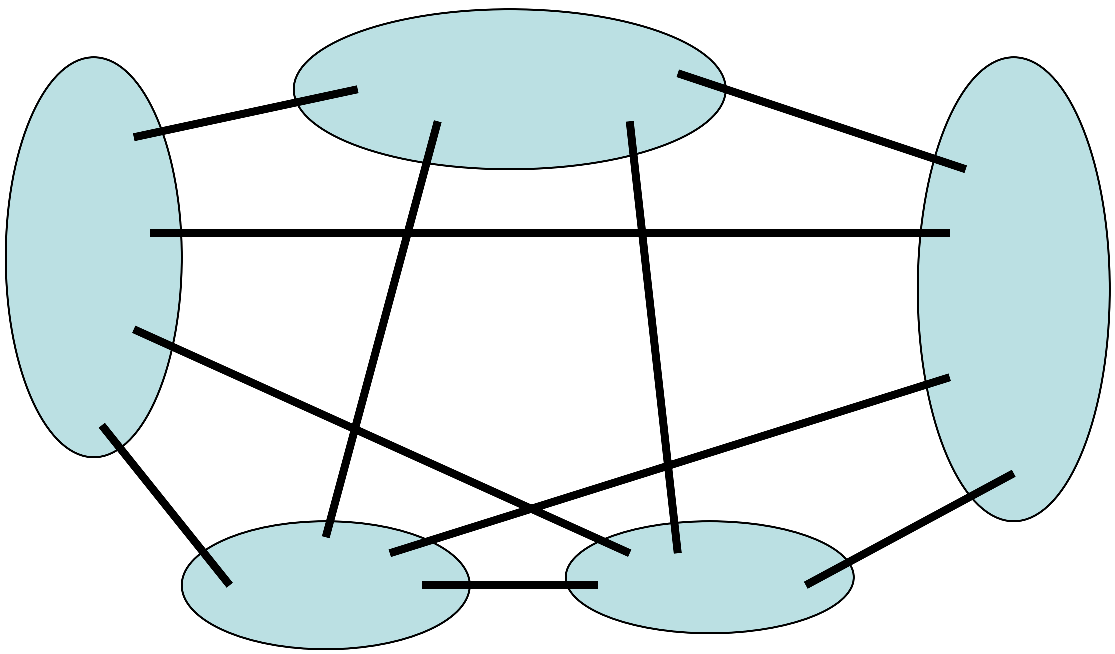
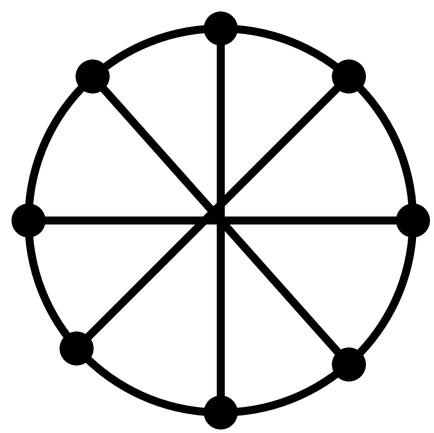
- A minor of G is obtained by taking subgraphs and contracting edges.
- Preserves planarity and other properties.
- G has an H minor ($H \leq_m G$) if G has a minor isomorphic to H.
- A K_5 minor:



Excluding K_t minors

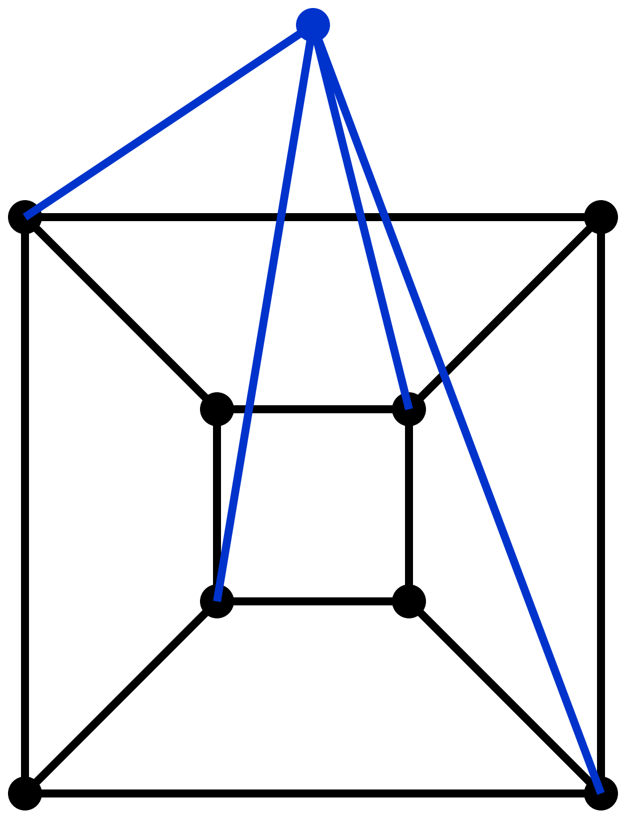
- $G \not\geq_m K_3 \Leftrightarrow G$ is a forest (tree-width ≤ 1)
- $G \not\geq_m K_4 \Leftrightarrow G$ is series-parallel (tree-width ≤ 2)
- $G \not\geq_m K_5 \Leftrightarrow$ tree-decomposition into planar graphs and V_8 (Wagner 1937)

- $G \not\geq_m K_6 \Leftrightarrow$???



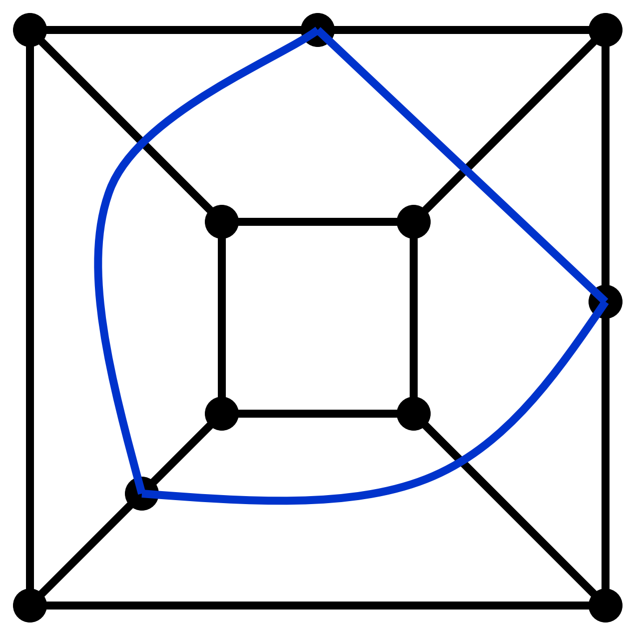
Graphs with no K_6

- apex ($G\backslash v$ planar for some v)



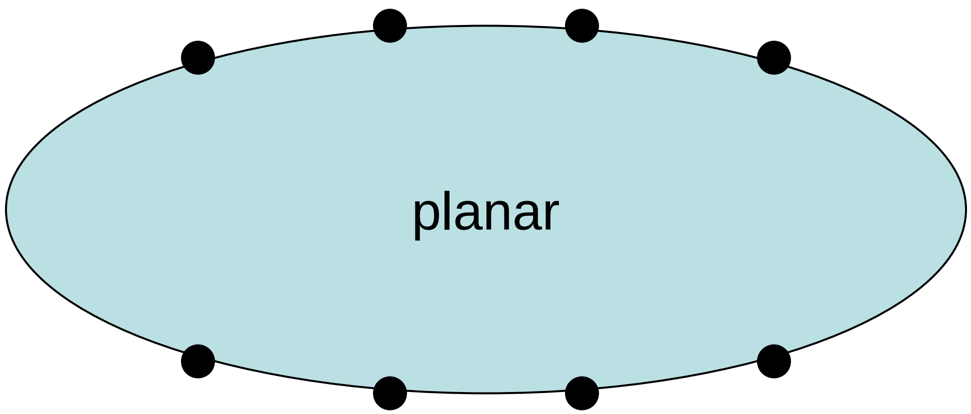
Graphs with no K_6

- apex ($G \backslash v$ planar for some v)
- planar + triangle



Graphs with no K_6

- apex ($G \backslash v$ planar for some v)
- planar + triangle
- double-cross



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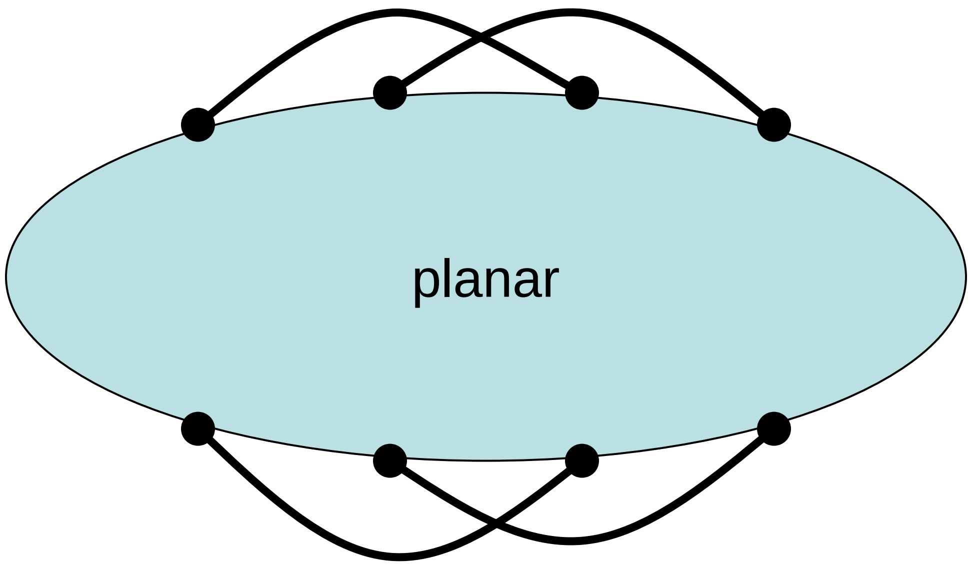

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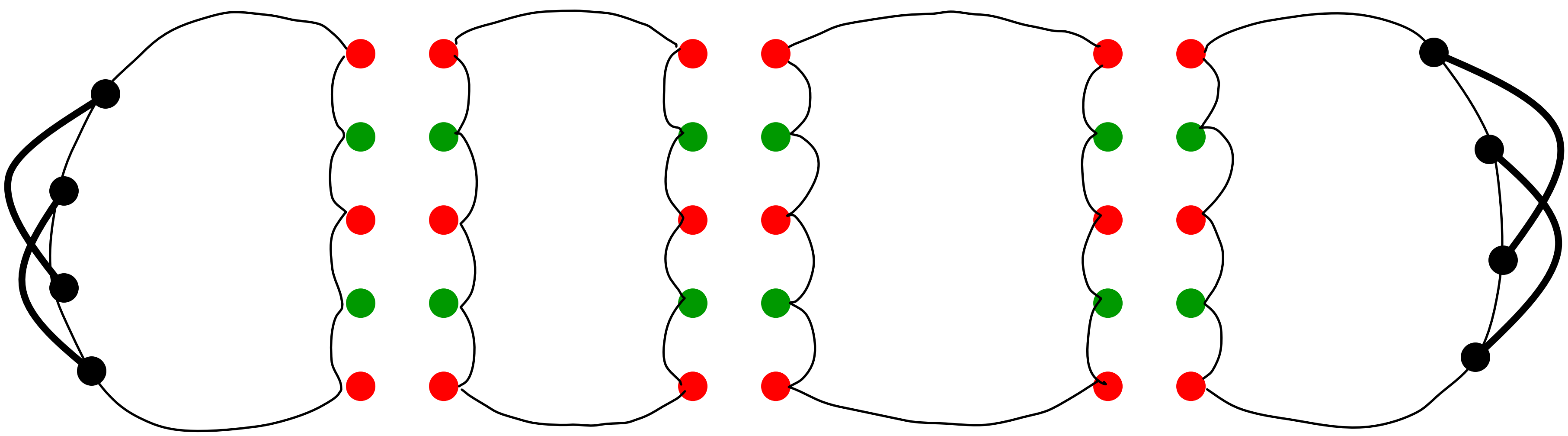

GRAPHS WITH NO K_t MINOR

REMARK

$G \not\geq_m K_t \Rightarrow$ (G + universal vertex) $\not\geq_m K_{t+1}$

REMARK

$G \backslash X$ planar for $X \subseteq V(G)$ of size $\leq t\text{-}5 \Rightarrow G \not\geq_m K_t$

GRAPHS WITH NO K_t MINOR

THEOREM (Robertson & Seymour)
$G \not\geq_m K_t \Rightarrow G$ has "structure"

Roughly structure means tree-decomposition of pieces that k-almost embed in a surface that does not embed K_t, where $k=k(t)$.

Converse not true, but:
G has "structure" $\Rightarrow G \not\geq_m K_{t'}$ for some $t' >> t$

Our objective is to find a simple iff statement

Extremal results for K_t

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So

- $G \not\succeq K_t \Rightarrow |\mathrm{E}(G)| \leq (t-2)n-(t-1)(t-2)/2$ for $t \leq 7$

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- $G \not\geq K_t \Rightarrow |E(G)| \leq ct(\log t)^{1/2}n$ (Kostochka, Thomason)

CONJ (Seymour, RT)
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CONJ (Seymour, RT) G is $(t\text{-}2)$-connected, big
$G \not\geq K_t \Rightarrow |E(G)| \leq (t\text{-}2)n\text{-}(t\text{-}1)(t\text{-}2)/2$

- $G \not\geq K_8 \Rightarrow |E(G)| \leq 6n\text{-}21$, unless G is a $(K_{2,2,2,2,2},5)$-cockade (Jorgensen)

- $G \not\geq K_9 \Rightarrow |E(G)| \leq 7n\text{-}28$, unless…. (Song, RT)

K_t minors naturally appear in:

Structure theorems:

-series-parallel graphs (Dirac)
-characterization of planarity (Kuratowski)
-linkless embeddings (Robertson, Seymour, RT)
-knotless embeddings (unproven)

Hadwiger's conjecture: $K_t \not\leq_m G \Rightarrow \chi(G) \leq t\text{-}1$

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Hadwiger's conjecture is open for $t>6$

Open even for G with no 3 pairwise non-adjacent vertices; HC implies any such $G \geq_m K_{\lceil n/2 \rceil}$

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True for big graphs:

There exists N such that every 6-connected graph $G \not\geq_m K_6$ on $\geq N$ vertices is apex.

MAIN THM (with Norin) $\forall\, t\ \exists\, N_t\ \forall\, t$-connected graph $G \not\geq_m K_t$ on $\geq N_t$ vertices $\exists\, X \subseteq V(G)$ with $|X| \leq t\text{-}5$ such that $G \backslash X$ is planar.

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NOTES

- Gives iff characterization
- t-connected and $|X| \leq t$-5 best possible
- N_t needed for $t>7$
- Proved for $31t/2$-connected graphs by Kawarabayashi, Maharry, Mohar

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INGREDIENTS IN THE PROOF

- "Brambles" ("tangles")
- Thm of DeVos-Seymour on graphs in a disk
- No big bramble $\Rightarrow$ bounded tree-width method
- Excluded K_t theorem of Robertson & Seymour to examine the structure of a big bramble

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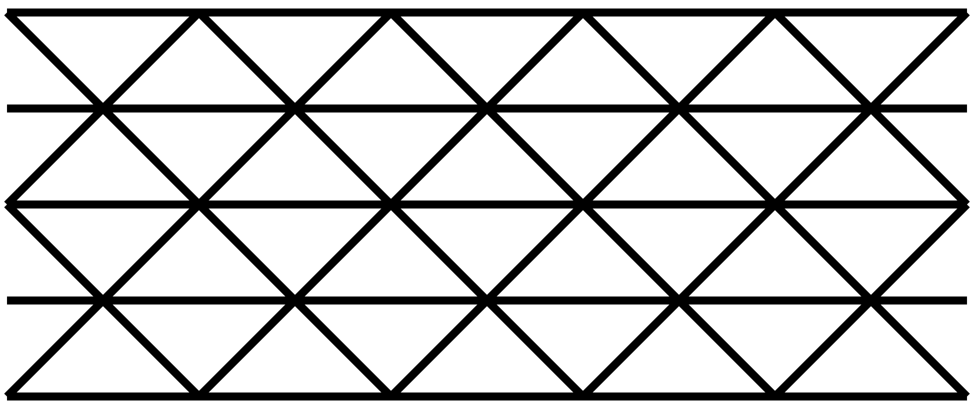

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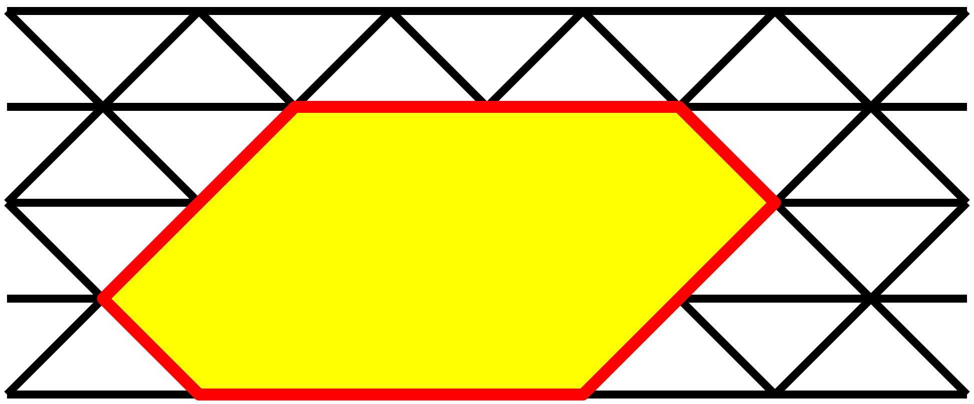
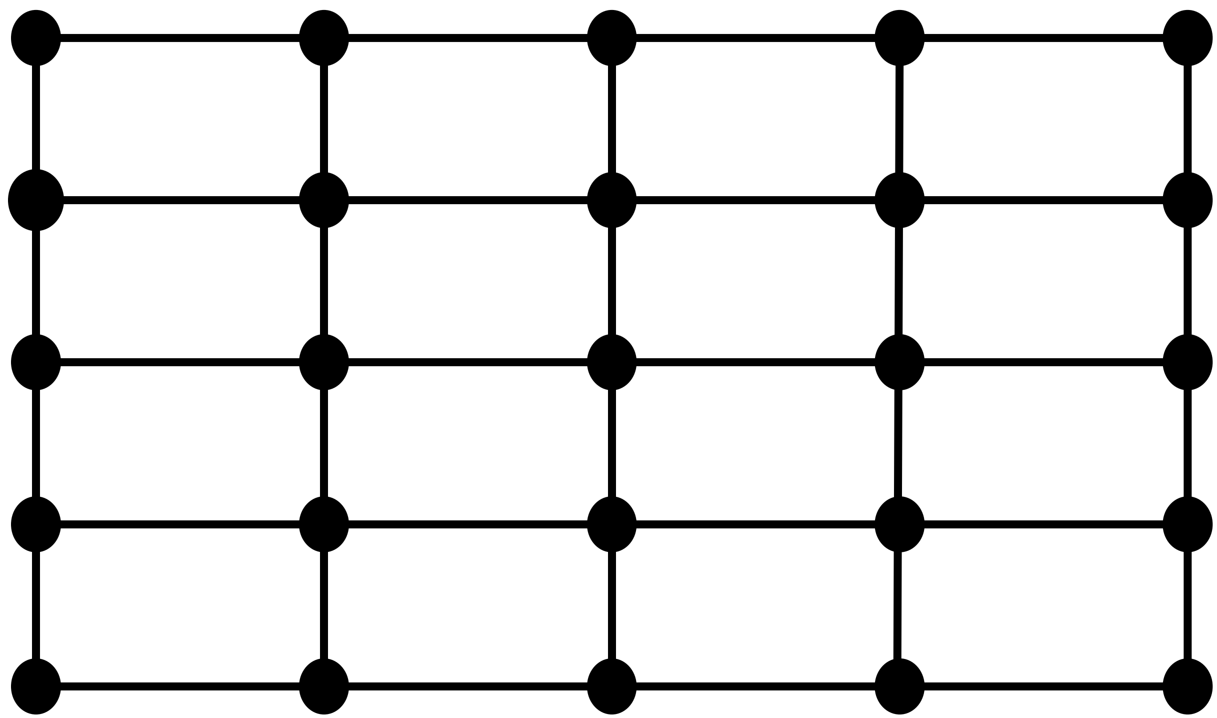


DEF A bramble $\mathcal{B}$ in G is a set of connected subgraphs that pairwise touch (intersect or are joined by an edge). The order of $\mathcal{B}$ is $\min\{|X| : X \cap B \neq \emptyset$ for every $B \in \mathcal{B}\}$.

EXAMPLE $G = k\mathrm{x}k$ grid, $\mathcal{B} = \{$all crosses$\}$, order is k



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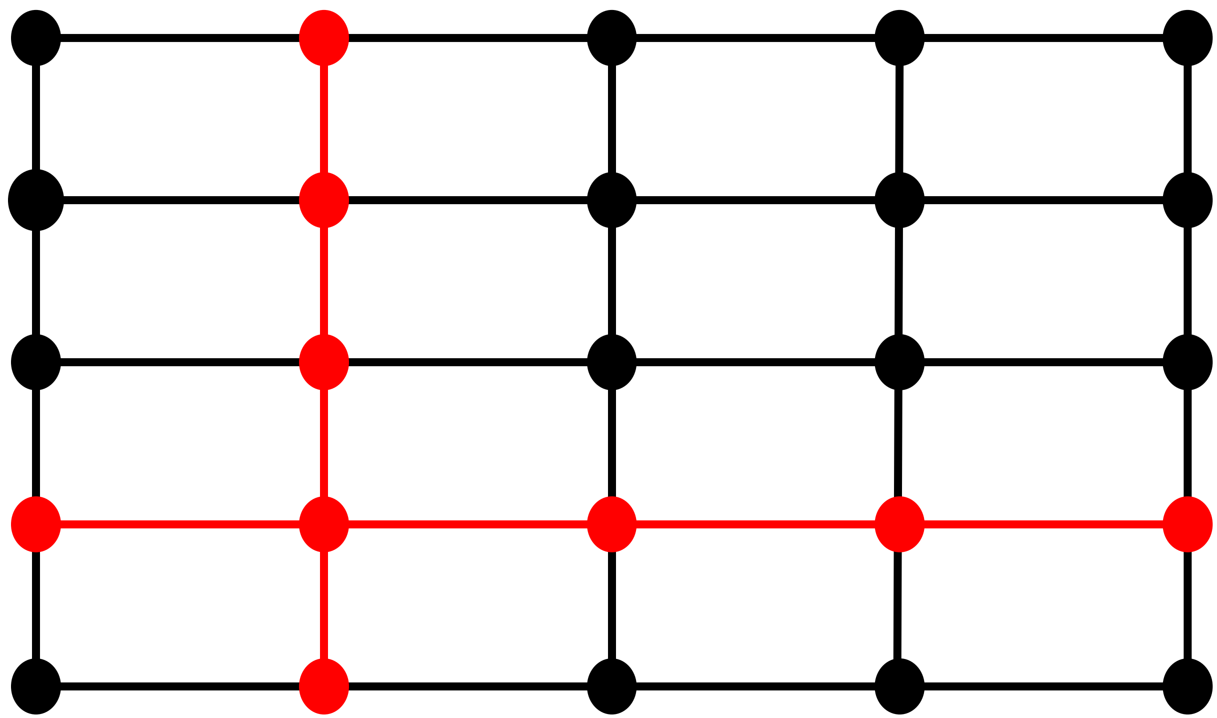

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THEOREM (Seymour, RT)
tree-width(G) = max order of a bramble + 1

THEOREM (Robertson, Seymour)
All brambles in G form a tree-decomposition.

CASE 1 G has bounded tree-width

PROOF Let (T,W) be a tree-decomposition of bounded width. T has a vertex of big degree or a long path.

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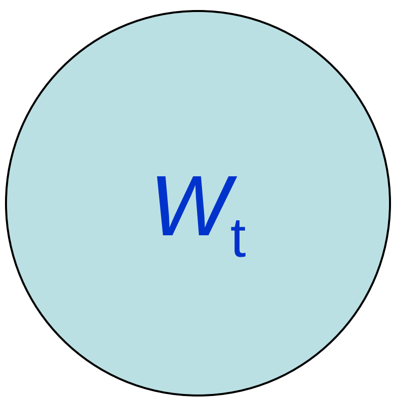

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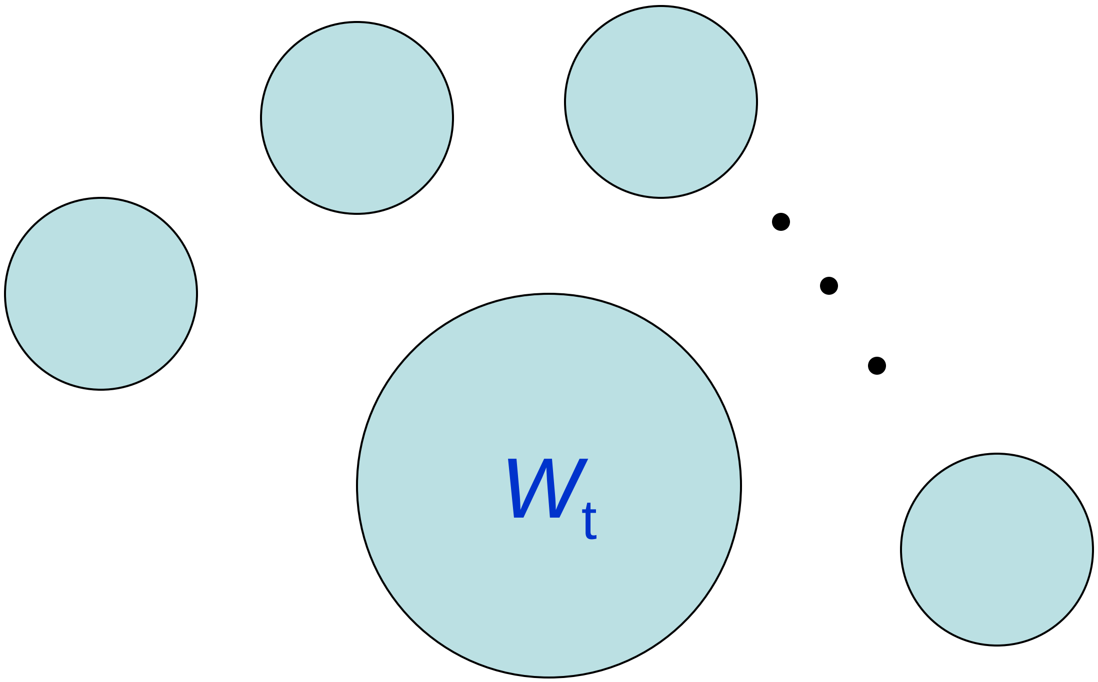

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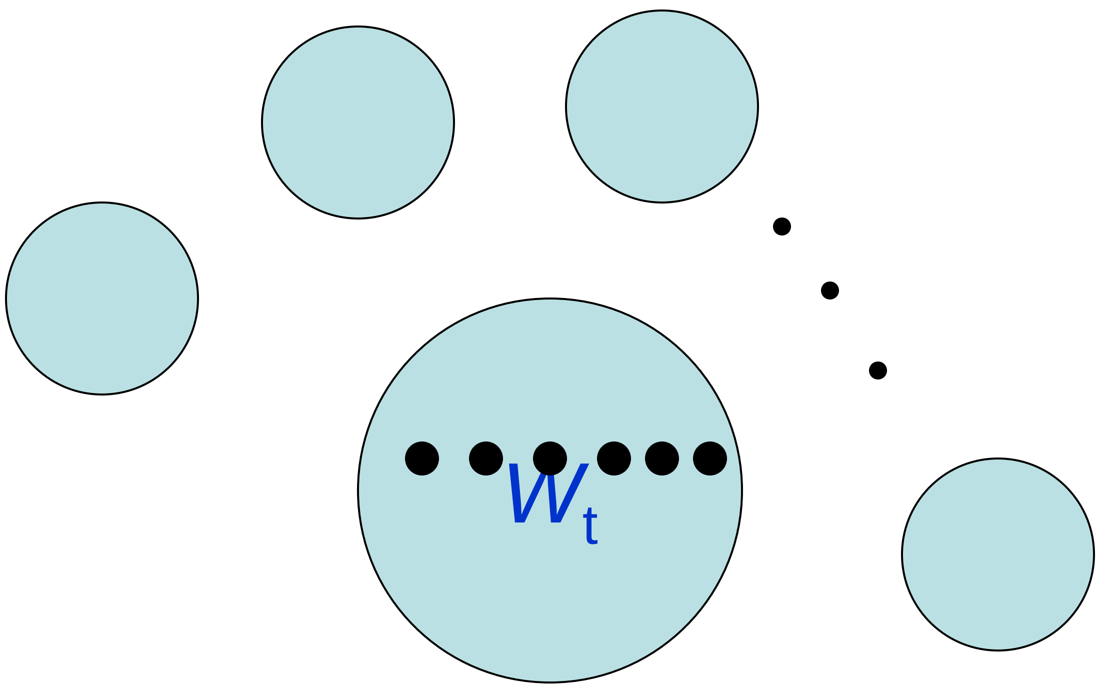

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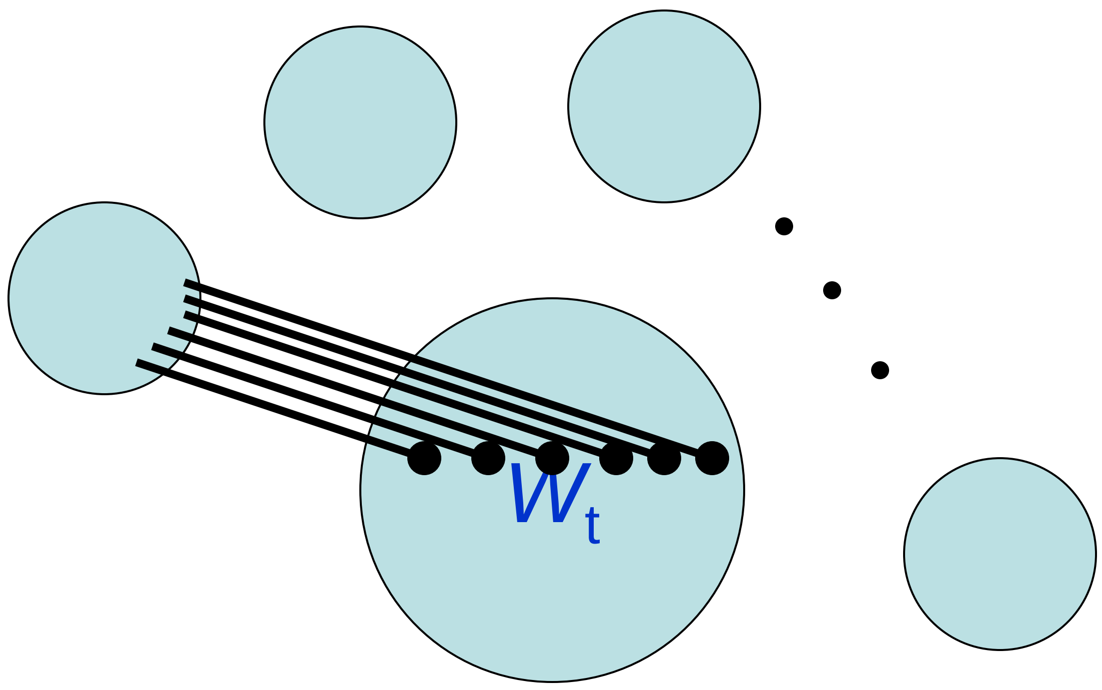

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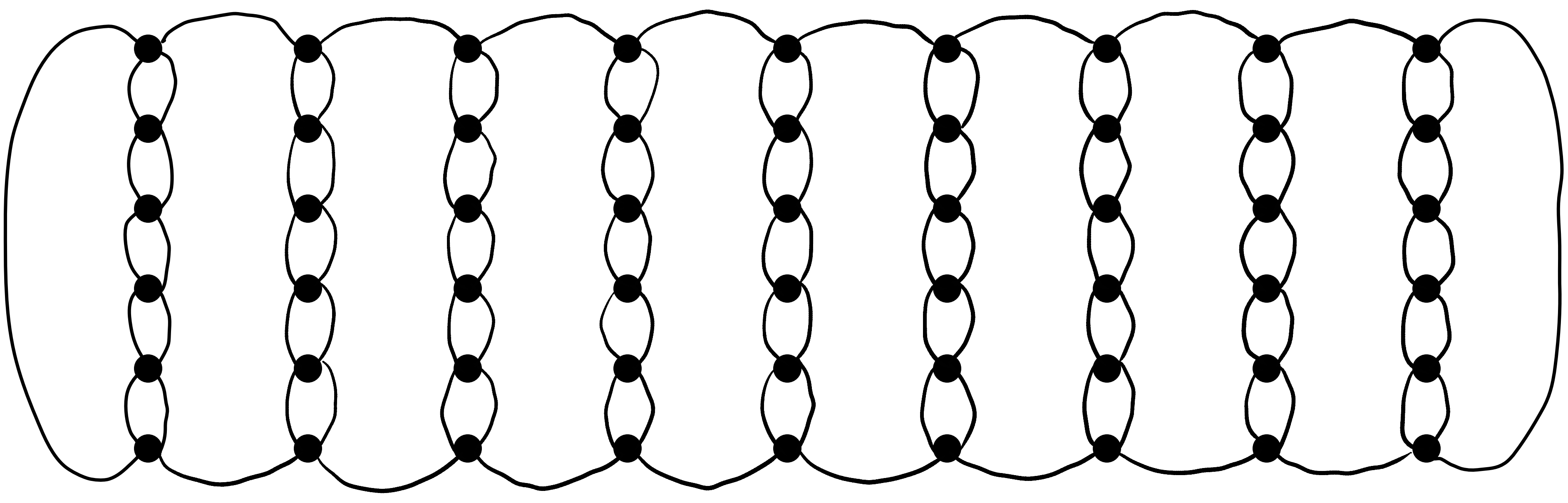

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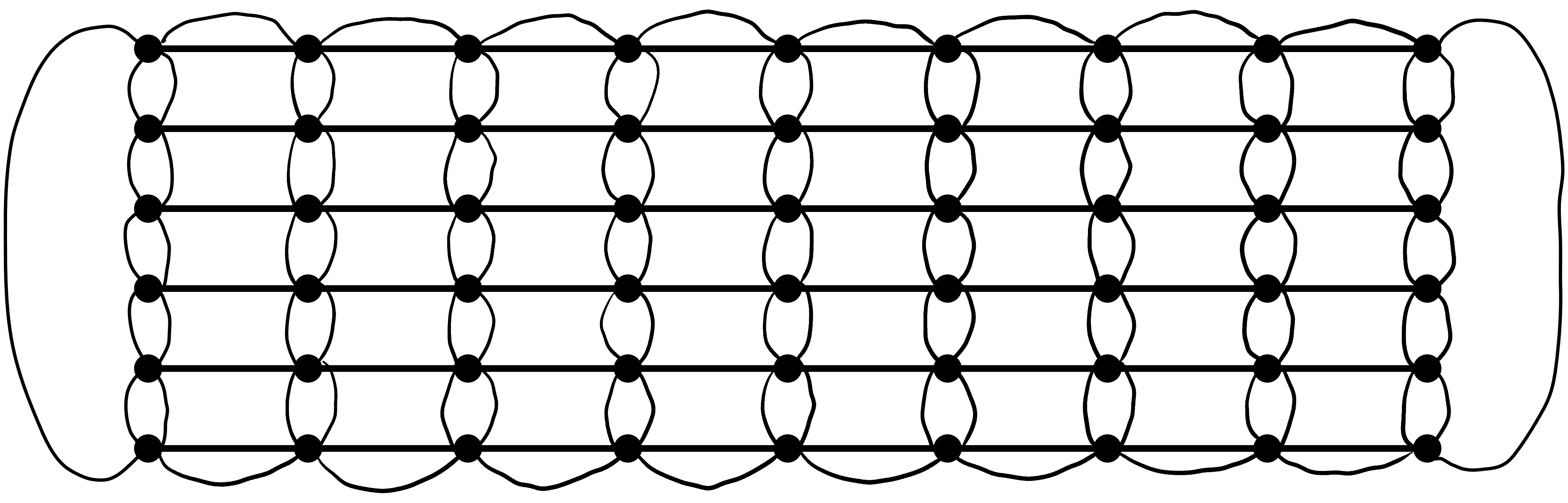

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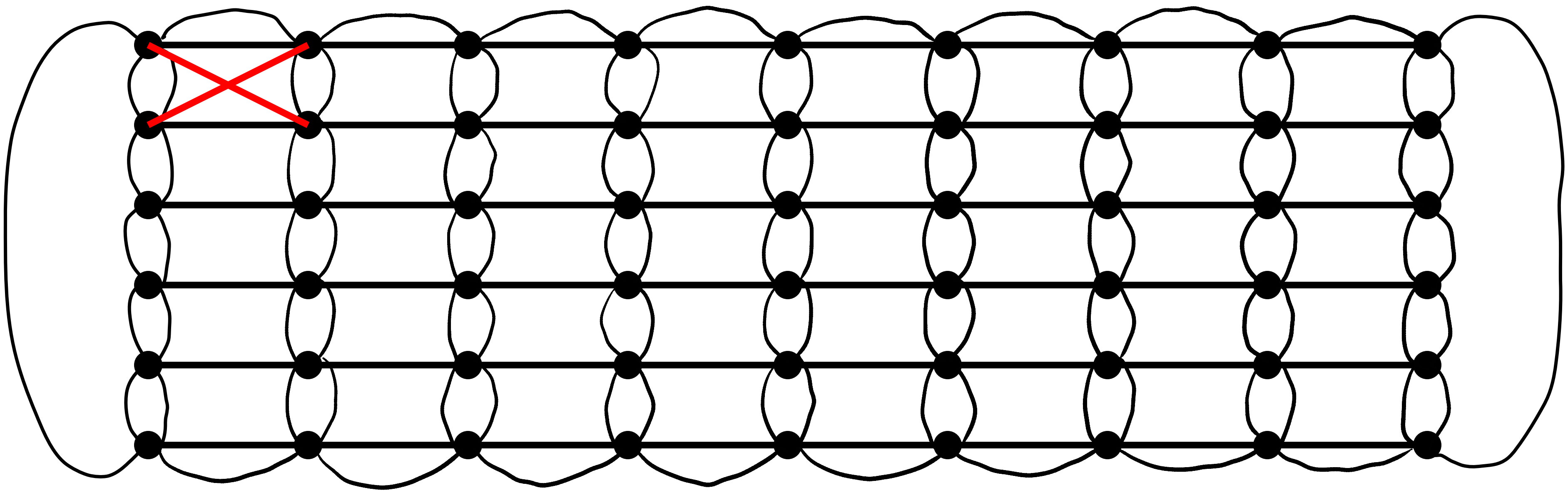

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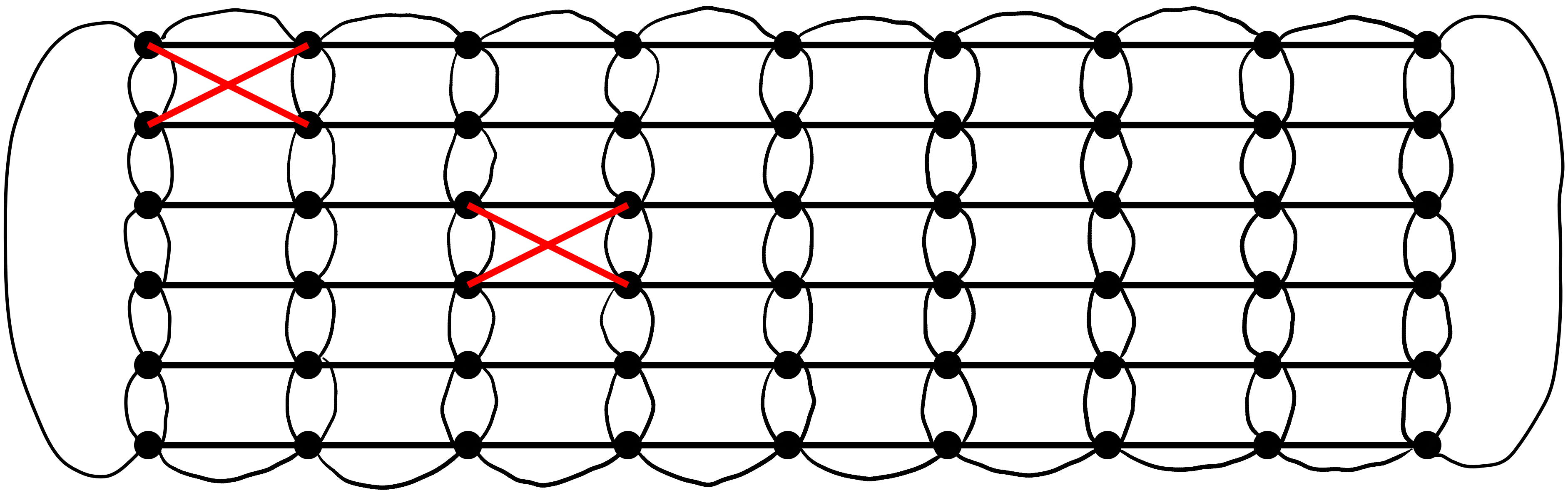

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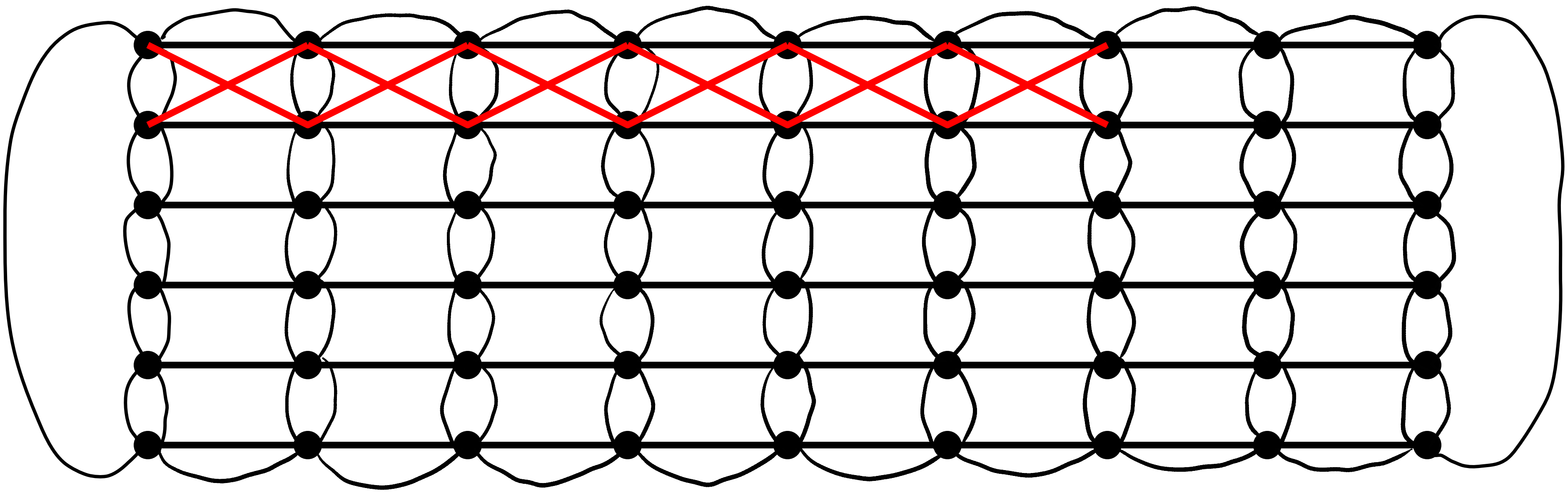

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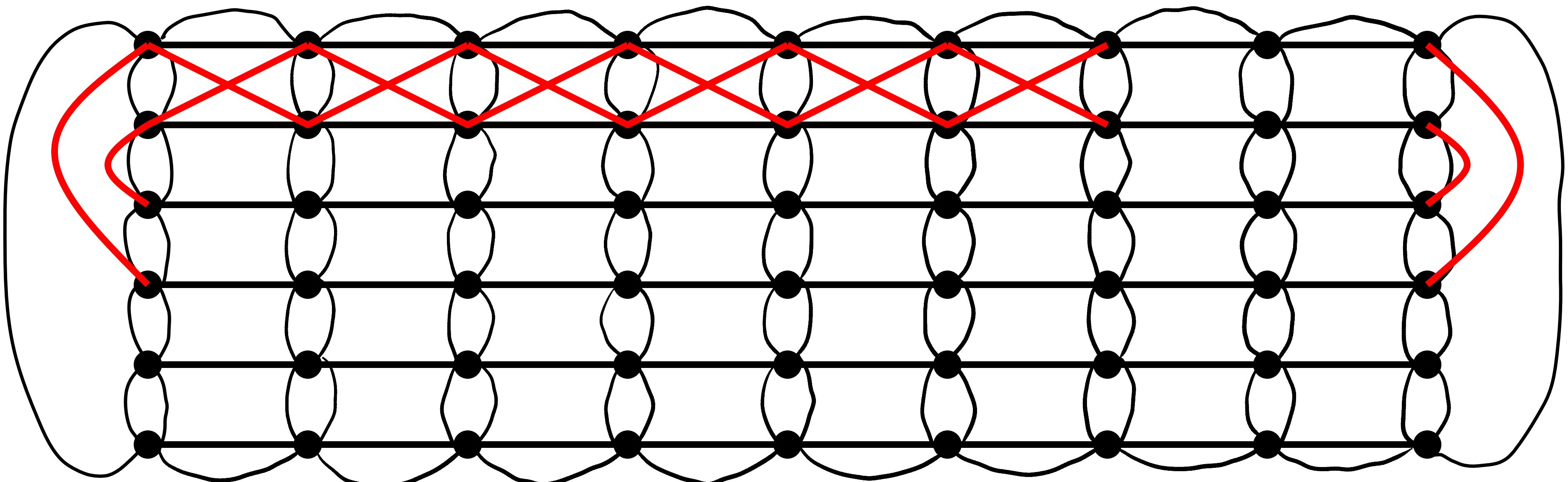

This suffices to get a K_7 minor. For bigger cliques we need a more sophisticated argument.

CASE 2 There is a bramble $\mathcal{B}$ of large order

By the excluded K_t theorem of Robertson and Seymour we reduce to the same problem as above.

SUMMARY

MAIN THM (with Norin) $\forall\ t\ \exists\ N_t\ \forall\ t$-connected graph $G \not\geq_m K_t$ on $\geq N_t$ vertices $\exists\ X \subseteq V(G)$ with $|X| \leq t$-5 such that $G \backslash X$ is planar.

COR G is t-connected, $\geq N_t$ vertices,
$G \not\geq_m K_t \Rightarrow |E(G)| \leq (t\text{-}2)n\text{-}(t\text{-}1)(t\text{-}2)/2$

CONJ Corollary holds for $(t\text{-}2)$-connected graphs