HADWIGER'S CONJECTURE FOR K_6-FREE GRAPHS

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HADWIGER'S CONJECTURE. Every loopless graph with no K_{t+1} minor is t-colorable.

•Trivial for $t \leq 2$, easy for $t = 3$ (Dirac)
•For $t = 4$ equivalent to the 4CT (Wagner)

THEOREM (Robertson, Seymour, RT)
Every minimal counterexample to Hadwiger's conjecture for $t = 5$ is apex ($G\backslash v$ is planar for some vertex v of G).

The above result does not assume the 4CT. By the 4CT every apex graph is 5-colorable and so no counterexample exists.

RELATED CONJECTURES

JORGENSEN'S CONJECTURE

Every 6-connected graph with no K_6-minor is apex.

HAJOS' CONJECTURE Every loopless graph with no K_{t+1} subdivision is t-colorable.

For $t \leq 3$ equivalent to Hadwiger ($\Rightarrow$ true)
$t = 4, 5, 6$ open
$t \geq 7$ false (Catlin)
In fact, the random graph is a counterexample.

THEOREM (Mader, conjectured by Dirac)
Every graph on n vertices and at least $3n-5$ edges has a K_5 subdivision.

CONJECTURE (Kelmans, Seymour)
Every 5-connected nonplanar graph has a K_5 subdivision.

Implies Mader's theorem (Kezdy, McGuiness)

Apex graph: $G\backslash v$ planar for some $v \in V(G)$

Hadwiger graph: counterexample to Hadwiger's conjecture for $t = 5$ with minimum number of vertices

THEOREM (R,S,T) Every Hadwiger graph is apex.

COROLLARY (assuming 4CT) No Hadwiger graph exists; that is, every graph with no K_6-minor is 5-colorable.

KNOWN RESULTS

THEOREM (Mader) Every Hadwiger graph is 6-connected.

THEOREM (Mader) If G has n vertices and no K_6-minor, then G has at most $4n-10$ edges.

For small t:

No K_t minor $\Rightarrow$ at most $(t-2)n - \binom{t-1}{2}$ edges

No K_2 minor $\Rightarrow$ at most 0 edges

No K_3 minor $\Rightarrow$ at most $n-1$ edges

No K_4 minor $\Rightarrow$ at most $2n-3$ edges

No K_5 minor $\Rightarrow$ at most $3n-6$ edges

No K_6 minor $\Rightarrow$ at most $4n-10$ edges

No K_7 minor $\Rightarrow$ at most $5n-15$ edges

No K_8 minor $\not\Rightarrow$ at most $6n-21$ edges

In general:

No K_t minor $\Rightarrow$ at most $O(t\sqrt{\log t}\,n)$ edges

Given G and $v_1, v_2, v_3, v_4 \in V(G)$, when can one contract a K_4 onto v_1, v_2, v_3, v_4?

LEMMA If G is 4-connected, then it can be done, unless G is planar with v_1, v_2, v_3, v_4 on the boundary of the same face.

COROLLARY No edge of a non-apex Hadwiger graph is in 4 triangles.

LEMMA Let G be a 4-connected non-planar graph, and let $v_1, v_2, v_3, v_4 \in V(G)$. Then there exists a K_4-minor on v_1, v_2, v_3, v_4.

PROOF

LEMMA (refinement) Let G be 2-connected, and let $v_1, v_2, v_3, v_4 \in V(G)$. If a K_4 minor cannot be contracted onto v_1, v_2, v_3, v_4 then

(i) G can be drawn in the plane with v_1, v_2, v_3, v_4 on the boundary of the infinite face, or

(ii) there is a (≤ 3)-separation (A, B) with $v_1, v_2, v_3, v_4 \in A$, $|B - A| \geq 2$ and $|\{v_1, v_2, v_3, v_4\} \cap B| \leq 2$, or

(iii) there is a "trisection" of order 2.

THEOREM Every Hadwiger graph is apex.

OUTLINE OF THE PROOF Let G be a non-apex Hadwiger graph.

STEP 1 G does not have three K_4 subgraphs pairwise meeting in ≤ 2 vertices.

STEP 2 G has minimum valency ≥ 7, except for at most two 6-valent vertices.

STEP 3 G is 7-connected, except for at most two 6-valent vertices.

STEP 4 G does not contain any of ten forbidden subgraphs.

STEP 5 G has a matching of cardinality $\lfloor \frac{1}{2}|V(G)| \rfloor$.

STEP 6 G does not exist.

STEP 1 G does not have three K_4 subgraphs pairwise meeting in ≤ 2 vertices.

COROLLARY Let us say that two K_4 subgraphs of G are equivalent if they meet in 3 vertices. Then

1. this is an equivalence relation,
2. each equivalence class has ≤ 2 members,
3. there are ≤ 2 equivalence classes,
4. there are ≤ 10 vertices in K_4-subgraphs.

STEP 2 G has minimum degree 7, except for at most two vertices of valency 6.

CLAIM If N is the set of neighbors of a vertex v of G of valency 6, then N does not contain an independent set of size 3.

By the claim N has a triangle, and so v is in a K_4-subgraph.

STEP 3 G is 7-connected, except for at most two vertices of valency 6.

Let J be the graph induced by the cutset. There are cases depending on J.

STEP 4 G does not contain any of 10 forbidden subgraphs.

STEP 5 G has a matching of cardinality $\lfloor \frac{1}{2}|V(G)| \rfloor$.

STEP 6 G does not exist

Assume for simplicity that $n = 2k = |V(G)|$. Then $|E(G)| \geq 7k - 1$. Let $e_1, e_2, \ldots, e_k$ be a matching, and let G' be obtained by contracting it. Then $|E(G')| \leq 4k - 10$ by Mader's theorem. Thus $3k + 9$ edges are lost, but only k edges were contracted. Where did the other edges go?