COLORING GRAPHS ON SURFACES

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Surface is a compact 2-dim manifold.

S_g: sphere $+$ g handles

N_k: sphere $+$ k cross-caps

Σ: S_g or N_k

THEOREM (Heawood)
The chromatic number of a graph on S_g is at most $(7 + \sqrt{48g + 1})/2$.

THEOREM (Thomassen) For every Σ, there are only finitely many 6-critical graphs in Σ.

COROLLARY For every Σ and every $k \geq 5$ there is a poly-time algorithm to decide if a graph in Σ is k-colorable.

REMARK NP-complete for $k = 3$, open for $k = 4$.

A CONSTRUCTION

Find a triangulation of Σ with all vertices of even degree. Add vertices and edges as indicated.

This gives a triangulation with exactly two vertices of odd degree. Those two vertices must receive the same color in any 4-coloring, and hence the graph is not 4-colorable.

HIGHER GIRTH

THEOREM (Fisk, Mohar) If G is a graph in S_g or N_g with no essential cycle of length $c \log g$ or less, then
if G has girth ≥ 4, then $\chi(G) \leq 4$
if G has girth ≥ 6, then $\chi(G) \leq 3$

REMARK (i) best possible for N_g.

THEOREM (Youngs) No quadrangulation of the projective plane has $\chi = 3$.

THEOREM (Gimbel, Thomassen)

A triangle-free graph in the projective plane is 3-colorable $\Leftrightarrow$ it has no subgraph that is a non-bipartite quadrangulation.

GRÖTZSCH'S THEOREM

THEOREM Every triangle-free planar graph is 3-colorable.

PROOF (Thomassen) Enough to show for graphs of girth at least five.

Identify diagonally
opposite vertices

One "reducible configuration":

Identify indicated
pairs of vertices

Stronger result for graphs of girth ≥ 5: Every 3-coloring of a facial cycle of length ≤ 8 extends to G.

CONJECTURE. For every integer k there exists an integer N such that if a 3-coloring of a facial k-cycle of a planar graph G of girth ≥ 5 does not extend into G, then it does not extend into some subgraph of G on at most N vertices.

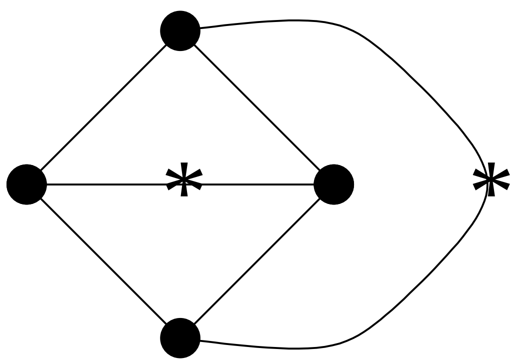
Walls found all "obstructions" for $k \leq 11$.

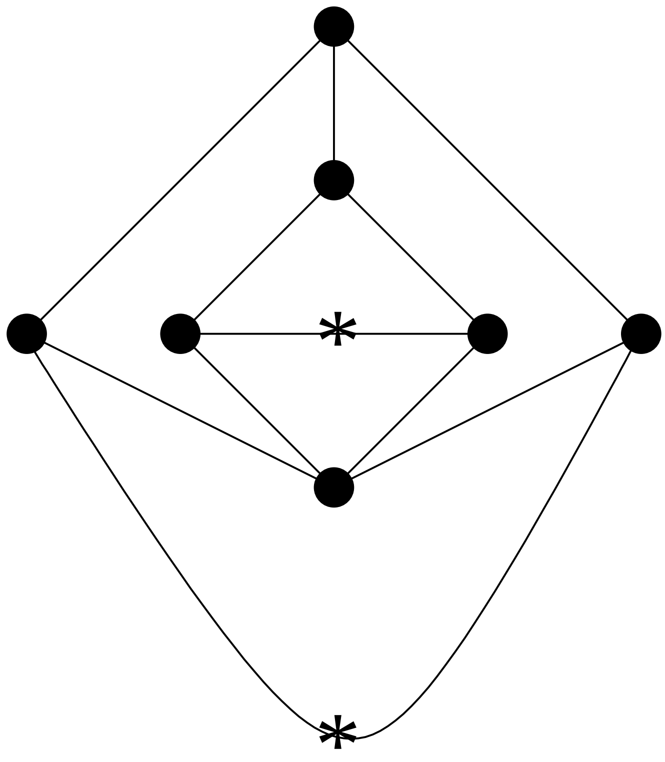
THEOREM (Thomassen) Every graph of girth ≥ 5 on the projective plane or the torus is 3-colorable.

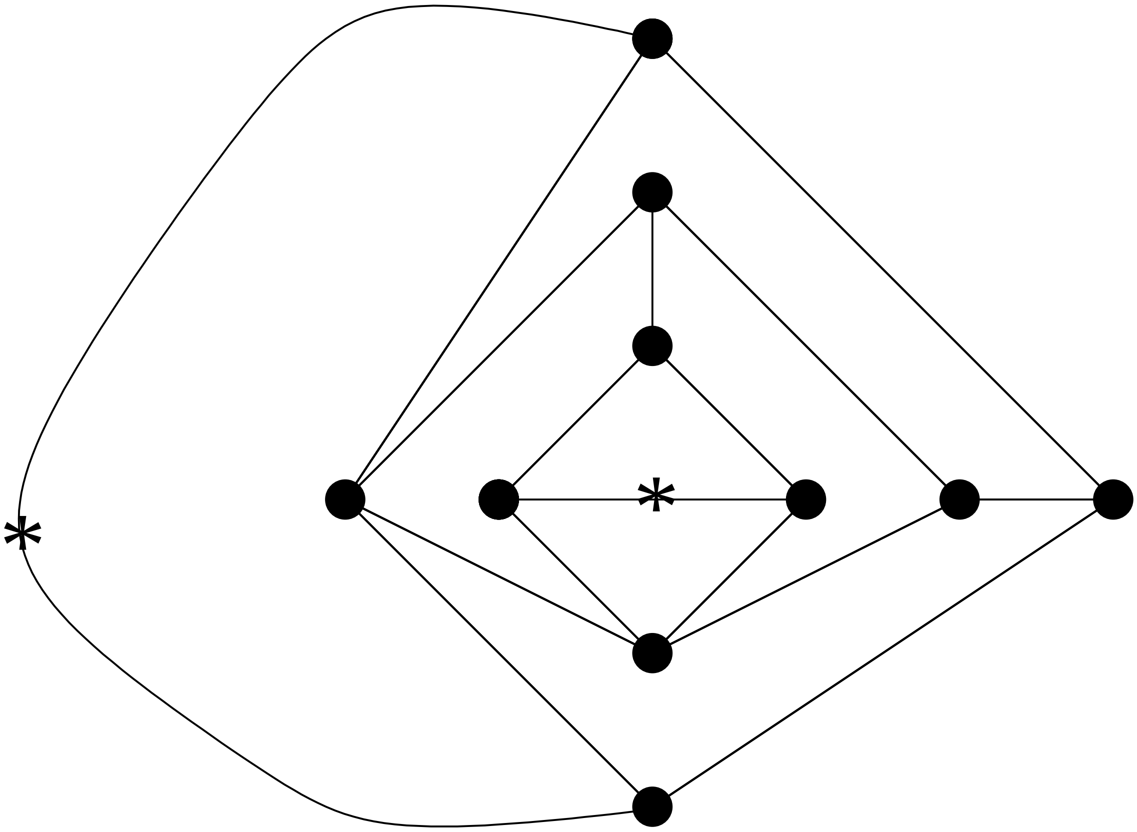
CONJECTURE For every Σ, there are only finitely many 4-critical graphs of girth ≥ 5 in Σ.

THEOREM (RT, Walls) There are none in the Klein Bottle.

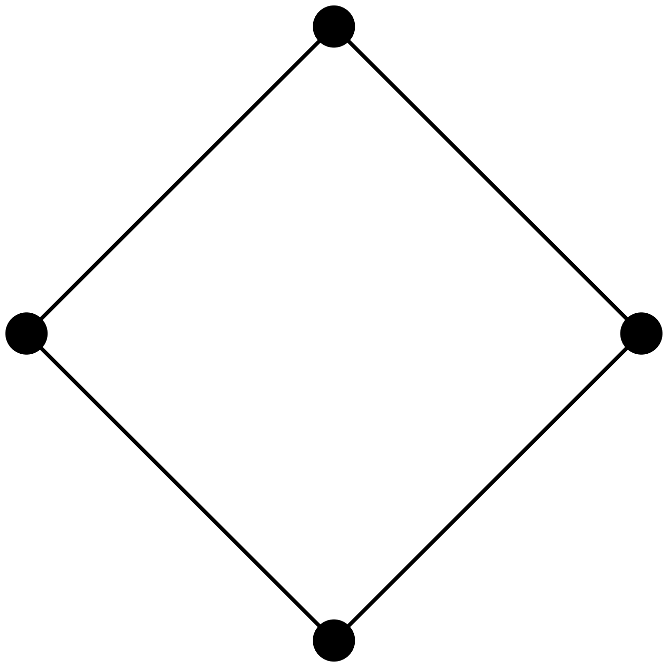
PROOF Let G be a graph in N_2 with no contractible cycles of length ≤ 4. Then G is 3-colorable $\Leftrightarrow$ no subgraph isomorphic to







One step: No minimal counterexample has a separating 4-cycle.



By induction if both sides have vertices, otherwise identify diagonally opposite vertices.

INDEPENDENT SETS

Every triangle-free planar or cubic graph on n vertices has an independent set of size $\geq n/3$. The constant is best possible, even for planar graphs with $\Delta \leq 4$.

THEOREM (Staton) $\Delta(G) \leq 3$, triangle-free $\Rightarrow \alpha \geq 5n/14.$

REMARK Best possible, because of $P(7,2)$.

PROOF (Heckman, RT) We prove $\alpha \geq \frac{1}{14}(5n+d-2\lambda)$, where $d = 3n-2e$ and λ is the number of "difficult" components:

The proof is by induction and gives a linear-time algorithm. Assume G is 3-regular.

By induction

$$\begin{aligned} \alpha(G') &\geq \tfrac{1}{14}(5(n-4)+6-2\lambda')+1 \\ &= \tfrac{1}{14}(5n-2\lambda') \end{aligned}$$

THEOREM (Heckman, RT, conjectured by Albertson, Bollobás and Tucker)
Every triangle-free planar graph on n vertices with $\Delta \leq 3$ has an independent set of size at least $3n/8$.

THEOREM (Heckman, RT)
$\mathcal{L}$-free $\Rightarrow$ $\alpha \geq 3n/8 + (3n-2e)/24 - \lambda/8$.

THEOREM Every triangle-free $\mathcal{L}$-free planar graph on n vertices and e edges with $\Delta \leq 3$ has an independent set of size at least $3n/8 + (3n - 2e)/24 - \lambda/8$.

$\chi^*(G)$, the fractional chromatic number of G, is the infimum of $\frac{a}{b}$, where
•each vertex assigned b colors from $\{1, .., a\}$
•adjacent vertices assigned disjoint sets.

CONJECTURE $\Delta(G) \leq 3$, triangle-free $\Rightarrow$ $\chi^*(G) \leq 14/5$.

CONJECTURE $\Delta(G) \leq 3$, triangle-free, planar $\Rightarrow$ $\chi^*(G) \leq 8/3$.

QUESTION G planar $\Rightarrow$ $\chi^*(G) \leq 4.99$.
Is there a computer-free proof?

THE FOUR COLOR THEOREM.

Every planar graph is 4-colorable.

Comments:

- Simple statement, yet proof is long.
- A 1976 proof by Appel and Haken.
- A simpler proof by Robertson, Sanders, Seymour, RT
- Proof is computer assisted.
- Over two dozen equivalent formulations (in terms of vector cross products, Lie algebras, divisibility, Temperley-Lieb algebras,...)
- Conjectured generalizations.
- See August 1998 Notices of the AMS for a survey.

THEOREM. (Matiyasevich) There exist linear functions A_k, B_k, C_k, D_k $(k = 1, .., 986)$ of 21 variables such that the 4CT is equivalent to the assertion that for every two integers n, m there exist integers $c_1, \ldots, c_{20}$ such that

$$\prod \binom{A_k(m, c_1, .., c_{20}) + 7^n B_k(m, c_1, .., c_{20})}{C_k(m, c_1, .., c_{20}) + 7^n D_k(m, c_1, .., c_{20})}$$

is not divisible by 7.

A graph H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges. An H minor is a minor isomorphic to H.

HADWIGER'S CONJECTURE. Every graph with no K_{t+1} minor is t-colorable.

• Trivial for $t \leq 2$, easy for $t = 3$ (Dirac)
• For $t = 4$ equivalent to the 4CT (Wagner)

THEOREM (Robertson, Seymour, RT) Every minimal counterexample to Hadwiger's conjecture for $t = 5$ is apex ($G \backslash v$ is planar for some vertex v of G).

The above result does not assume the 4CT. By the 4CT every apex graph is 5-colorable and so no counterexample exists.

A k-flow in a graph G is an integer flow f in an orientation of G with $0 < |f(e)| < k$ for every edge e of G.

CONJECTURE (Tutte 1972) Every edge 4-connected graph has a 3-flow.

CONJECTURE (Tutte 1966) Every edge 2-connected graph with no Petersen minor has a 4-flow.

CONJECTURE (Tutte 1954) Every edge 2-connected graph has a 5-flow.

EDGE 3-COLORING

THEOREM (Tait) The 4CT is equivalent to the statement that every 2-connected cubic planar graph is edge 3-colorable.

THEOREM (Robertson, Sanders, Seymour, RT) Every 2-connected cubic graph with no Petersen minor is edge 3-colorable.

MAIN STEPS IN THE PROOF

(1) Every minimal counterexample is apex or doublecross.
(2) True for apex graphs.
(3) True for doublecross graphs.

THEOREM (Thomson, RT) Every edge 2-connected graph with no $P_{10}\backslash e$ (Petersen minus an edge) minor has a 4-flow.

THEOREM (Thomson, RT) Every internally 4-connected nonplanar graph with minimum degree ≥ 3 and girth ≥ 5 has a $P_{10}\backslash e$ minor.

THEOREM (Thomson, RT) Assume that G is nonplanar, has girth ≥ 5, minimum degree ≥ 3 and every vertex that belongs to a pentagon has degree 3. Then G has a minor isomorphic to Petersen, Triplex, Box, or Ruby.

OPEN PROBLEMS

(1) Minor-minimal graphs of girth ≥ 5 and minimum degree ≥ 3

(2) Minor-minimal 5-connected graphs

(3) Is it true that if a 5-connected graph G has a minor in the Petersen family, then every member of the Petersen family is a minor of G?