MATHEMATICAL INDUCTION

Mathematical induction is a technique for proving statements about the natural numbers. The following is an example of what can be proved by induction.

Example 1. Prove that for every integer $n \geq 1$,

$$1 + 2 + \cdots + n = n(n+1)/2.$$

Let us state the principle of mathematical induction, and then explain how to use it.

THE PRINCIPLE OF MATHEMATICAL INDUCTION. *Let q be an integer (for example $q = 1$) and let $S(n)$ be a statement about the integer n (for example $S(n)$ can be the statement "$1 + 2 + \cdots + n = n(n+1)/2$"). To prove that $S(n)$ is true for all integers $n \geq q$ (which is what we are after in the above example) it is sufficient to prove the following:*

(1) $S(q)$ is true, and

(2) if $k \geq q$ is an integer and $S(k)$ is true then $S(k+1)$ is true.

Step (1) is called the *basis step*, step (2) is called the *induction step*. Thus in order to prove that $S(n)$ is true for all integers $n \geq q$ we need to verify both the basis step and the induction step. Here is how it works.

Example 1 (solution). Take $q = 1$ (because we want to prove something for all $n \geq 1$) and take $S(n)$ to be the statement "$1 + 2 + \cdots n = n(n+1)/2$". We must show that

(i) $S(1)$ is true (the basis step), and that

(ii) if $k \geq 1$ and $S(k)$ is true, then $S(k+1)$ is true.

Basis step. $S(1)$ is true, because $S(1)$ says that $1 = 1 \times 2/2$, which is obviously true.

Induction step. Let $k \geq 1$ be an integer and assume that $S(k)$ is true. We must show that $S(k+1)$ is true. In other words:

We know that $1 + 2 + \cdots + k = k(k+1)/2$ (this is $S(k)$), and we must show that $1 + 2 + \cdots k + k + 1 = (k + 1)(k + 2)/2$.

To show this last statement we proceed as follows:

$$1 + 2 + \cdots + k + k + 1 = (1 + 2 + \cdots + k) + k + 1 = k(k + 1)/2 + k + 1$$
$$= (k + 1)(k/2 + 1) = (k + 1)(k + 2)/2.$$

The first equality is trivial (we have just inserted parentheses), in the second equality we have replaced the expression in parenthesis by the right–hand–side from the "we know" line above, and the rest is just an elementary algebra. This finishes the induction step.

We have thus verified both the basis step and the induction step, and therefore the statement follows from the Principle of Mathematical Induction.

Remark. The content of the "we know" line above is called the *induction hypothesis*. In other words it is the assumption that $S(k)$ is true in the induction step. Go back and see where the induction hypothesis was used in the above proof.

Example 2. Prove that for every $n \geq 1$,

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Solution. Take $q = 1$ and $S(n)$ to be the statement "$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$." Apply mathematical induction.

Basis step. $\frac{1}{1\times 2} = \frac{1}{2}$, and so the basis step is true.

Induction step. Let $k \geq 1$.

We know (this is the induction hypothesis) $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$.

We must show $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$.

We have

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$
$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)}$$
$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Again, the first equality is trivial, for the second we use the induction hypothesis, and the rest is algebra.

Example 3. Prove that for all $n \geq 2$,

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{13}{24}$$

Solution. We take $q = 2$ and $S(n)$ to be the statement "$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{13}{24}$."

Basis step. $\frac{1}{3} + \frac{1}{4} > \frac{13}{24}$, which is true, because $\frac{7}{12} > \frac{13}{24}$.

Induction step. Let $k \geq 2$.

We know $\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k} > \frac{13}{24}$.

Must show that $\frac{1}{(k+1)+1} + \frac{1}{(k+1)+2} + \cdots + \frac{1}{2(k+1)} > \frac{13}{24}$.

We have

$$\frac{1}{(k+1)+1} + \frac{1}{(k+1)+2} + \cdots + \frac{1}{2(k+1)} = \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2} = \frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1} > \frac{13}{24} + \frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1}$$

The first two equalities are easy. Notice the trick in the second — we have added and subtracted $\frac{1}{k+1}$ in order to obtain the left–hand–side of the induction hypothesis. The last inequality above is from the induction hypothesis.

Unfortunately, we are not done yet. It is not clear whether we have made progress. But we have! If we could show that $\frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1} \geq 0$ then we will have shown that the last expression above is $\geq \frac{13}{14}$, and hence that $\frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2(k+1)} > \frac{13}{14}$, which is what we want. So let us try to prove that

$$\frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1} \geq 0. \tag{1}$$

Since $k \geq 2$ we can multiply both sides by $2(2k+1)(k+1)$ and the inequality remains unchanged. We get

$$2(k+1) + (2k+1) - 2(2k+1) \geq 0$$

which is true (the left–hand–side equals 1). Since this last inequality holds it follows that (1) holds as well and therefore we are done, because we argued earlier that it suffices to prove (1).

Problems

1. Prove that for all $n \geq 1$, $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$.
2. Prove that for all $n \geq 1$, $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$.
3. Prove that for all $n \geq 1$, $1 + 4 + 7 + \cdots + (3n-2) = n(3n-1)/2$.
4. Prove that for all $n \geq 1$, $1^2 + 3^2 + \cdots + (2n-1)^2 = n(2n-1)(2n+1)/3$.
5. Prove that for all $n \geq 1$, $1^3 + 3^3 + \cdots + (2n+1)^3 = (n+1)^2(2n^2+4n+1)$.
6. Prove that for all $n \geq 1$, $1 \times 2 + 2 \times 3 + \cdots + n(n+1) = n(n+1)(n+2)/3$.
7. Prove that for all $n \geq 1$, $1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + n(n+1)(n+2) = n(n+1)(n+2)(n+3)/4$.
8. Prove that for all $n \geq 1$, $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{n+1}\right) = \frac{1}{n+1}$.
9. Prove that for all $n \geq 1$, $1 \times 1! + 2 \times 2! + \cdots + n \times n! = (n+1)! - 1$.
10. Prove that for all $n \geq 1$, $\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$.