1 Dimension of lin($\mathcal{M}$)

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Exercise 1.1 (Homework Due Feb 7) Construct a graph G and a tight cut C that is not of any of types listed among our examples of tight cuts.

Conjecture 1.2 *Let D be a digraph and A a subset of $V(D)$. An A-cycle is a dicycle C in D such that $V(C) \cap A \neq \emptyset \neq V(C) - A$. Is there a function $f : N \to N$ such that, for any k and D, D has either k disjoint A-cycles, or a set $X \subset V(D)$ of size $\leq f(k)$ such that $D - X$ has no A-cycles.*

Exercise 1.3 There exists a function $g : N \to N$ such that, for any graph G and any k, G has either k disjoint cycles, or a set $X \subset V(G)$ of size $\leq g(k)$ such that $G - X$ has no cycles. Moreover, $g(k) = \Theta(k \log k)$.

Recall the linear hull lin($\mathcal{M}$) is the same as $\{x \in R^{E(G)} : x(C) = x(D), \forall \text{ tight cuts } C, D\}$, where $\mathcal{M}$ is the set of induced vectors of perfect matchings.

Lemma 1.4 *Let A be the incidence matrix of a connected graph G. Then*

$$\begin{aligned} \text{rank}(A) &= n - 1, \text{ if } G \text{ is bipartite,} \\ &= n, \text{ otherwise,} \end{aligned}$$

where $n = |V(G)|$.

Let T be the incidence matrix of tight cuts vs edges. We observed

$$\begin{aligned} \text{aff}(\mathcal{M}) &= \{x \in R^{E(G)} : x(C) = 1 \text{ for } \forall \text{tight cut } C\} = \{x : Tx = \mathbf{1}\}, \\ \dim \text{aff}(\mathcal{M}) &= m - \text{rank}(T), \end{aligned}$$

where $m = |E(G)|$.

Corollary 1.5

$$\begin{aligned} \dim \text{lin}(\mathcal{M}) &= \dim \text{aff}(\mathcal{M}) + 1 \\ &= m - \text{rank} T + 1. \end{aligned}$$

Theorem 1.6 *If G is a brick, then* $\dim \text{lin}(\mathcal{M}) = m - n + 1$.

Proof.

$$\begin{aligned}\dim\operatorname{lin}(\mathcal{M}) &= m - \operatorname{rank}(T) + 1 = m - \operatorname{rank}(A) + 1 \\ &= m - n + 1\end{aligned}$$

□

Lemma 1.7 *If G is bipartite, then* $\dim\operatorname{lin}(\mathcal{M}) = m - n + 2$.

Proof. In a bipartite graph, the rows of T corresponding to trivial cuts generate the row space of T. In other words, the characteristic vector of a tight cut is a linear combination of characteristic vectors of trivial tight cuts.

Let (A, B) be the bipartition, and $X \subset B$ and $Y \subset A$ form a tight cut $C = \delta(X \cup Y)$. Then we have $|X| + 1 = |Y|$ and $C = \delta(Y) - \delta(X)$, or $|Y| + 1 = |X|$ and $C = \delta(X) - \delta(Y)$, and so, by symmetry, we may assume the former. Assign $+1$ to X-vertices and -1 to Y-vertices, and notice that

$$\mathbf{1}_C = \sum_{y \in Y} \mathbf{1}_{\delta(y)} - \sum_{x \in X} \mathbf{1}_{\delta(x)}.$$

Thus,

$$\begin{aligned}\dim\operatorname{lin}(\mathcal{M}) &= m - \operatorname{rank}(T) + 1 \\ &= m - \operatorname{rank} A + 1 \\ &= m - (n-1) + 1 = m - n + 2\end{aligned}$$

□

Definition 1.8 Let $C = \delta(S)$ be a cut in a graph G. Let G_1 be obtained by identifying all vertices in S into a new vertex and let G_2 be obtained by identifying $V(G) - S$ into a new vertex. We call G_1, G_2 the two C-contractions of G.

Note that $E(G_1) \cup E(G_2) = E(G)$ and $E(G_1) \cap E(G_2) = C$.

Lemma 1.9 *Let G be 1-extendable, C a tight cut, and let G_1, G_2 be the two C-contractions of G. Then*

$$\dim\operatorname{lin}(\mathcal{M}(G)) = \dim\operatorname{lin}(\mathcal{M}(G_1)) + \dim\operatorname{lin}(\mathcal{M}(G_2)) - |C|.$$

Proof. Let $\mathcal{F}_i \subset \mathcal{M}(G_i)$ be a basis for $\mathrm{lin}(\mathcal{M}(G_i))$, and $\mathcal{F}_i^e = \{F \in \mathcal{F}_i : e \in F\}$ for $e \in C$, $i = 1, 2$. Fix a perfect matching F_i^e in $\mathcal{F}_i^e$ for every $e \in C$.

It suffices to show the claim that

$$\bigcup_{e \in C} \{F \cup F_2^e : F \in \mathcal{F}_1^e\} \cup \{F_1^e \cup F : F \in \mathcal{F}_2^e\}$$

is a basis for $\mathrm{lin}(\mathcal{M}(G))$. Note that $F_1^e \cup F_2^e$ are counted twice.

To prove the claim, let's show linear independence first. Suppose

$$\sum_{e \in C} \sum_{F \in \mathcal{F}_1^e} \lambda_F \mathbf{1}_{F \cup F_2^e} + \sum_{e \in C} \sum_{F \in \mathcal{F}_2^e - \{F_2^e\}} \mu_F \mathbf{1}_{F_1^e \cup F} = \mathbf{0}. \tag{1}$$

Restricted to $E(G_1)$ this gives

$$\sum_{e \in C} \sum_{F \in \mathcal{F}_1^e} \lambda_F \mathbf{1}_F + \sum_{e \in C} \left(\sum_{F \in \mathcal{F}_2^e - \{F_2^e\}} \mu_F \right) \mathbf{1}_{F_1^e} = \mathbf{0}. \tag{2}$$

Then $\lambda_F = 0$ for all $F \in \mathcal{F}_1^e - \{F_1^e\}$. From (1) restricted to $E(G_2)$, we get

$$\sum_{e \in C} \left(\sum_{F \in \mathcal{F}_1^e} \lambda_F \right) \mathbf{1}_{F_2^e} + \sum_{e \in C} \sum_{F \in \mathcal{F}_2^e - \{F_2^e\}} \mu_F \mathbf{1}_F = \mathbf{0},$$

which implies $\mu_F = 0$ for all $F \in \mathcal{F}_2^e - \{F_2^e\}$. Now (2) implies $\lambda_F = 0$ for all $F \in \mathcal{F}_1$. So we have the linearity.

It is a simple exercise that the set is spanning. □

Exercise 1.10 Prove that the set is spanning.

By the tight cut decomposition of a graph G, we mean repeatedly replacing G by the two C-contractions of G for some tight cut C. At the end we end up with a list of bricks and braces. Those are called the *bricks and braces of* G.

Theorem 1.11 *(Lovasz) The underlying simple graphs of the bricks and braces resulting from a tight cut decomposition do not depend on the choice of tight cuts made during the process.*

Proof. omitted.

Definition 1.12 A *brick* of a graph G is any brick B obtained at the end of a tight cut decomposition of G.

Theorem 1.13 *Let G be a connected 1-extendable graph. Then* $\dim \mathrm{lin}(\mathcal{M}) = m - n + 2 - b$*, where b is the number of bricks in a tight cut decomposition of G.*

Proof. By induction on $E(G)$. If G has no tight cut, then it is a brick or a brace, and the theorem follows from earlier results. Thus we may assume that G has a tight cut, say C. Let G_1 and G_2 be the two C-contractions of G, and let b_i be the number of bricks in a tight cut decomposition of G_i. By Theorem 1.6, Lemma 1.7, Lemma 1.9 and the induction hypothesis,

$$\begin{aligned}
\dim \lim(\mathcal{M}(G)) &= \dim \lim(\mathcal{M}(G_1)) + \dim \lim(\mathcal{M}(G_2)) - |C| \\
&= m(G_1) - n(G_1) + 2 - b_1 \\
&+ m(G_2) - n(G_2) + 2 - b_2 \\
&- |C|.
\end{aligned}$$

Note $m(G_1) + m(G_2) - |C| = m(G)$ and $n(G) = n(G_1) + n(G_2) - 2$. Then, we have

$$\dim \lim(\mathcal{M}(G)) = m(G) - n(G) + 2 - b_1 - b_2 = m(G) - n(G) + 2 - b,$$

as desired. □