# LECTURES ON PERFECT GRAPHS

Lectures by
**Robin Thomas**
written down by his students

A graph is *perfect* if for every induced subgraph, the chromatic number is equal to the maximum size of a complete subgraph. Perfect graphs appear naturally in a variety of circumstances, such as communication complexity, integrality of polyhedra, polyhedral combinatorics, the channel assignment problem, the geometric algorithms of Grötschel, Lovász and Schrijver, or semi-definite programming. They are also the source of fundamental mathematical problems.

The course will cover classical topics in perfect graph theory, and will go through selected portions of the recent proof of the Strong Perfect Graph Conjecture, with emphasis on open problems and areas for future research.

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# 1 Introduction

Graphs are finite and simple; that is, they have no loops or multiple edges. Let $G$ be a graph. A *hole* in $G$ is an induced cycle of length at least four. An *antihole* in $G$ is an induced subgraph isomorphic to the complement of a hole. (The *complement* of a graph $G$ is the graph with vertex-set $V(G)$ and edge-set consisting precisely of all distinct pairs of vertices that are not adjacent in $G$.) A *clique* in $G$ is a set $X \subseteq V(G)$ of pairwise adjacent vertices, and a *stable set* is a set of pairwise non-adjacent vertices. The size of a maximum clique of a graph $G$ is denoted by $\omega(G)$ and the size of a maximum stable set is denoted by $\alpha(G)$. The *chromatic number* of $G$, denoted by $\chi(G)$, is the least number of colors needed to color the vertices of $G$ in such a way that adjacent vertices receive different colors. In other words, $\chi(G)$ is the minimum number $k$ such that the vertex-set $V(G)$ of $G$ can be partitioned into $k$ stable sets. Clearly $\chi(G) \geq \omega(G)$ for every graph $G$, but equality need not hold. For instance, if $G$ is an odd hole (i.e., a hole with an odd number of vertices), then $\chi(G) = 3 > 2 = \omega(G)$. Similarly, if $G$ is an odd antihole, say on $2k+1$ vertices, then $\chi(G) = k+1 > k = \omega(G)$.

A graph $G$ is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph $H$ of $G$; that is, the chromatic number of $H$ is equal to the maximum size of a clique of $H$. Thus odd holes and odd antiholes are not perfect, and neither is any other graph that has an odd hole or an odd hole antihole. Are there any other imperfect graphs? Berge [1] conjectured in 1961 that there are not, and a proof of that will be discussed later.

**1.1** *A graph is perfect if and only if it has no odd hole and no odd antihole.*

We will begin by discussing the relevance of perfect graphs. We refer the reader to the excellent book by Ramírez Alfonsín and Reed [11] for further reading.

# 2 Examples of perfect graphs

A graph $B$ is *bipartite* if its vertex-set $V(B)$ can be partitioned into two disjoint sets $X, Y$ in such a way that every edge of $B$ has one end in $X$ and the other end in $Y$. In other words, $B$ is bipartite if and only if $\chi(B) \leq 2$. It follows immediately that every bipartite graph is perfect.

Now let $G$ be the complement of a bipartite graph $B$, in symbols $G = \overline{B}$. We claim that $G$ is perfect. Since every induced subgraph of $G$ is the complement of an appropriate subgraph of $B$, it suffices to show that $\chi(G) = \omega(G)$. But $\omega(G) = \alpha(B)$, the maximum size of a stable set of $B$, and $\chi(G)$ is equal to the minimum number of cliques of $B$ covering the vertices of $B$. Thus the statement $\chi(G) = \omega(G)$ is equivalent to the assertion that the vertices of the bipartite graph $B$ can be covered by $\alpha(B)$ edges and vertices. The latter is a classical result of König [6] and can be found in almost every graph theory textbook.

Another two classes of perfect graphs can be obtained from bipartite graphs by means of line graphs. Let $G$ be a graph. The *line graph* of $G$ is the graph $L(G)$ with vertex-set $E(G)$ in which $e, f \in E(G)$ are adjacent if they share an end in $G$. Now if $B$ is a bipartite graph, then $G = L(B)$ is perfect. To see this, it again suffices to show that $\chi(G) = \omega(G)$. Now $\omega(G)$ is equal to the maximum degree of $B$, and $\chi(G)$ is equal to the *edge-chromatic number* of $B$, the minimum number of colors needed to color the *edges* of $B$ in such a way that adjacent edges receive different colors. The latter two numbers are equal in any bipartite graph $B$ by a well-known theorem of König [6].

Similarly, if $B$ is a bipartite graph, then $G = \overline{L(B)}$, the complement of $L(B)$, is perfect. To see this it again suffices to show that $\chi(G) = \omega(G)$. But $\chi(G)$ is the minimum number of cliques of $L(B)$ that cover the vertices of $L(B)$, which in turn is equal to the minimum number of vertices of $B$ that meet all the edges of $B$. On the other hand, $\omega(G)$ is the size of a maximum stable set in $L(B)$, which is equal to the maximum size of a matching of $B$. But the maximum size of a matching in a bipartite graph is equal to the minimum number of vertices meeting all the edges of $B$ by another classical theorem of Egerváry [3] and König [7].

Thus we have seen four classes of perfect graphs. The perfection of the first class is easy, and for the other three classes their perfection is equivalent to an old and well-known result in graph theory. By now many more classes have been discovered; by the last count there are 96 known classes of perfect graphs.

# 3 The perfect graph theorem

Notice that in the above examples of perfect or imperfect graphs it was always the case that a graph is perfect if and only if so is its complement. This is not a coincidence, due to the following theorem of Lovász [8], originally conjectured by Berge [1].

**3.1 (Perfect Graph Theorem)** *A graph is perfect if and only if its complement is perfect.*

For the proof we will need two lemmas. The first is easy.

**3.2** *A graph $G$ is perfect if and only if every induced subgraph $H$ of $G$ has a stable set that intersects every maximum clique of $H$.*

The second lemma we need is due to Lovász [8].

**3.3 (Replication Lemma)** *Let $G$ be a perfect graph, and let $v \in V(G)$. Define a graph $G'$ by adding a new vertex $v'$ and joining it to $v$ and all the neighbors of $v$. Then $G'$ is perfect.*

**Proof.** By (3.2) it suffices to prove that $G'$ itself has a stable set that intersects every maximum clique of $G'$. Since $G$ is perfect, it has a coloring using $\omega(G)$ colors; let $S$ be the color class containing $v$. Then $S$ is as desired.
□

In the proof of (3.1) we will give another characterization of perfect graphs in terms of certain polytopes associated with graphs. Let $G$ be a graph. The *stable set polytope* of $G$, also known as the *vertex packing polytope* of $G$, denoted by STAB($G$), is the convex hull in $\mathbf{R}^{V(G)}$ of all incidence vectors of stable sets of $G$. A related polytope is the *fractional stable set polytope* or *fractional vertex packing polytope* QSTAB($G$) $\subseteq \mathbf{R}^{V(G)}$ defined by the constraints

$$x_v \geq 0 \quad \text{for every } v \in V(G), \tag{1}$$

$$\sum_{v \in V(K)} x_v \leq 1 \qquad \text{for every clique } K \text{ in } G. \tag{2}$$

Since the incidence vector of every stable set satisfies (1) and (2), we have $\mathrm{STAB}(G) \subseteq \mathrm{QSTAB}(G)$. The following theorem, which also incorporates results of Chvátal [2] and Fulkerson [4], [5] implies the Perfect Graph Theorem. Our treatment follows [10].

**3.4** *For any graph $G$, the following conditions are equivalent.*

(i) *$G$ is perfect,*

(ii) *$STAB(G)$ =$QSTAB(G)$,*

(iii) *$\overline{G}$ is perfect,*

(iv) *$STAB(\overline{G})$ =$QSTAB(\overline{G})$.*

**Proof.** (i) $\Rightarrow$ (ii). Let $x$ satisfy (1) and (2). By rational approximation we may assume that $x$ is rational. Take an integer $N$ such that $y = Nx$ is an integer vector. By (1), $y_v \geq 0$ for all $v \in V(G)$. Let $Y_v$ ($v \in V(G)$) be disjoint sets with $|Y_v| = y_v$, and let $H$ be the graph with vertex–set $\bigcup_{v\in V(G)} Y_v$ in which a vertex of $Y_v$ is adjacent to a vertex of $Y_u$ if and only if either $v = u$ or $v$ is adjacent to $u$ in $G$. By a repeated application of the Replication Lemma (3.3) the graph $H$ is perfect. By (2) we have $\omega(H) \leq N$, and hence $V(H)$ can be covered by $N$ disjoint stable sets of $H$. Let $q_1, q_2, \ldots, q_N$ be the incidence vectors of the corresponding stable sets of $G$. Then $x = \frac{1}{N}(q_1 + q_2 + \cdots + q_N)$, as desired.

(ii) $\Rightarrow$ (iii) Since condition (ii) is inherited by induced subgraphs of $G$, by (3.2) it suffices to prove that $G$ has a clique that intersects every maximum stable set of $G$. We prove this assertion by induction on $|V(G)|$. We may assume that every vertex belongs to a maximum stable set of $G$, for otherwise we may delete such vertex and proceed by induction. Let $X$ be the subset of STAB($G$) consisting of all vectors $x$ with $\sum_{v\in V(G)} x_v = \alpha(G)$. Then $X$ is a face of QSTAB($G$), and hence one of the inequalities (1), (2) is satisfied with equality for every vector in $X$. Since every vertex of $G$ belongs to a maximum stable set of $G$, it follows that none of the inequalities (1) is satisfied with equality for all $x \in X$, and hence one of the inequalities (2) is. Thus there exists a clique of $G$ which meets every maximum stable set of $G$, as desired.

The remaining two implications follow by applying the previous arguments to the complement of $G$. □

# 4 Integrality of polyhedra

Let $A$ be a $0,1$-matrix and consider the following linear program:

$$\max c \cdot x \text{ subject to } x \geq 0 \text{ and } Ax \leq 1. \tag{3}$$

For which matrices $A$ is it true that for every objective function $c$, the linear program has integral optimum solution? This is an important question, because solving integer programs is an NP-hard problem, whereas efficient algorithms exist to solve linear programs. It turns out that the answer to our question leads directly to perfect graphs.

We say that the $i$th row of a matrix $A = (a_{ij})$ is *undominated* if there is no row index $j \neq i$ such that $a_{il} \leq a_{jl}$ for all $l$. Let $G$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, and let $K_1, K_2, \ldots, K_m$ be its (inclusion-wise) maximal cliques. We define the *maximal clique versus vertex incidence matrix* of $G$ to be the $m \times n$ matrix $A = (a_{ij})$, where $a_{ij} = 1$ if $v_j \in K_i$, and $a_{ij} = 0$ otherwise. The following is a result of Chvátal [2].

**4.1** *The linear program* (3) *has an integral optimum solution for every objective function c if and only if the undominated rows of A form the maximal clique versus vertex incidence matrix of a perfect graph.*

**Proof.** To prove "only if" let $A = (a_{ij})$ be an $m \times n$ matrix, and assume that (3) has an integral optimum for every nonnegative objective function $c$. We may assume that $A$ has no dominated rows (by deleting them). Let $G$ be the graph with vertex-set $\{1, 2, \ldots, n\}$ in which $i$ is adjacent to $j$ if and only if $a_{li} = a_{lj} = 1$ for some row index $l$. We first show that $A$ is a maximal clique versus vertex incidence matrix of $G$. Indeed, if that were not the case, then $G$ has a clique $K$ such that no row of $A$ has ones in the positions corresponding to the vertices of $K$ and possibly elsewhere. Choose such $K$ with $|K|$ minimal. Then clearly $|K| \geq 3$ by construction. Let $c$ be the incidence vector of $K$. By the minimality of $K$, for every $v \in K$ there exists a row with ones in columns corresponding to $K - \{v\}$. This implies that $c \cdot x \leq 1$ for every integral vector $x \in \mathbf{R}^{V(G)}$ satisfying $x \geq 0$ and $Ax \leq 1$. Now let $y \in \mathbf{R}^{V(G)}$ be the vector whose coordinates are $\frac{1}{|K|-1}$ in positions corresponding to elements of $K$, and all other coordinates are 0. Then $Ay \leq 1$, and yet $c \cdot y = \frac{|K|}{|K|-1} > 1$, a contradiction. This proves that $A$ is a maximal clique versus vertex incidence matrix of $G$, and hence the polyhedron $\{x \in \mathbf{R}^{V(G)} : x \geq 0 \text{ and } Ax \leq 1\}$ is precisely QSTAB($G$). Now the integrality of optima implies that every vertex of QSTAB($G$) is integral, and hence is a characteristic vector of a stable set. Thus STAB($G$) = QSTAB($G$), and hence $G$ is perfect by (3.4).

To prove the "if" part, let $G$ be perfect. We may assume that $A$ itself is the maximal clique versus vertex incidence matrix of $G$. By (3.4) the linear program (3) is equivalent to maximizing $c \cdot x$ subject to $x \in$ STAB($G$), which clearly attains its optimum at a characteristic vector of a stable set, and hence has an integral optimum solution, as desired. □

## 5 Berge's motivation: Shannon capacity

Let us go back in history and review what motivated Berge to introduce perfect graphs. Let us consider the transmission of symbols from some finite *input alphabet* $\Sigma$ through a discrete memoryless channel. Certain symbols may be confused during transmission, and we are interested in how many $n$-symbol error-free messages there are.

To be more precise, with every $a \in \Sigma$ we associate symbols $a_1, a_2, \ldots$ of some *output alphabet*. The meaning of this is that when $a$ is sent through the channel, one of $a_1, a_2, \ldots$ is received, say according to some probability distribution. However, in this model we are interested in transmissions that are 100% error-free, and so we define confoundability as follows. Let $a \in \Sigma$ be as above, and let $b \in \Sigma$ have $b_1, b_2, \ldots$ as the corresponding members of the output alphabet. We say that $a$ and $b$ are *confoundable* if $a_i = b_j$ for some $i, j$, and otherwise we say that they are *unconfoundable*. Finally, we say that two words $x, y \in \Sigma^t$ of length $t$ are *unconfoundable* if for at least one coordinate $i$ the corresponding entries $x_i$ and $y_i$ are unconfoundable.

As an example, let $\Sigma = \{a, b, c, d, e\}$, where the confoundable pairs are precisely $ab, bc, cd, de, ea$. In that case the symbols $a, c$ may be sent without danger of confusion, and hence there are at least $2^n$ $n$-symbol error-free messages obtained by taking all $n$-element sequences of the symbols $a$ and $c$. But we can do better. The 2-symbol messages $ab, bd, ca, dc, ee$ are pairwise unconfoundable (for instance $a$ may be confused with $b$, but $b$ cannot be confused with $d$, and hence the pairs $ab$ and $bd$ are unconfoundable), and by forming arbitrary words composed of those two-letter words we see that there are at least $5^{\lfloor n/2 \rfloor}$ $n$-symbol error-free messages, an improvement over the earlier bound. An asymptotic study of the bit-per-symbol error-free transmission rate leads to the notion of Shannon capacity of a graph [12], as follows.

Let $G$ be the graph with vertex-set $\Sigma$ and edge-set all pairs of unconfoundable elements of $\Sigma$. Thus we are interested in maximum cliques of $G^t$, where $G^t$ is the graph with vertex-set all $t$-tuples of vertices of $G$ in which two such $t$-tuples are adjacent if and only if for some coordinate the corresponding entries are adjacent in $G$. The Shannon capacity of $G$ is defined as

$$\Theta(G) := \lim_{n\to\infty} \frac{1}{n} \log \omega(G^n).$$

To see that the limit exist let $a_t := \log(\omega(G^t))$. Then

$$\begin{aligned} \omega(G^r)\cdot\omega(G^s) &\le \omega(G^{r+s}) \\ \log(\omega(G^r)) + \log(\omega(G^s)) &\le \log(\omega(G^{r+s})) \\ a_r + a_s &\le a_{r+s}. \end{aligned}$$

**Exercise 5.1** Given a sequence $\{a_i\}_{t=1}^{\infty}$ of non-negative numbers such that $a_r + a_s \le a_{r+s}$ for all $r, s$, prove that $\lim_{t\to\infty} \frac{a_t}{t}$ exists (or is $+\infty$) and is equal to $\sup_{t\ge 1} \frac{a_t}{t}$.

**Definition 5.2** For given $G, H$, define a graph $G \otimes H$ by $V(G\otimes H) = V(G)\times V(H)$ and $(g_1, h_1) \sim (g_2, h_2)$ if $g_1 \sim g_2$ or $h_1 \sim h_2$ (or both).

**Observation 5.3**

$$\omega(G)\cdot\omega(H) \le \omega(G\otimes H).$$

**Observation 5.4**

$$\chi(G)\cdot\chi(H) \ge \chi(G\otimes H).$$

Hence

$$\omega^n(G) \le \omega(G^n) \le \chi(G^n) \le \chi^n(G).$$

If $\omega(G) = \chi(G)$, then $\Theta(G) = \log(\omega(G)) = \log(\chi(G))$. This raises the question of what are the minimal graphs that do not satisfy $\chi(G) = \omega(G)$, and that lead Berge to formulate his influential conjectures.

We now turn to a celebrated theorem of Lovász [9].

**Theorem 5.5** $\Theta(C_5) = \frac{1}{2}\log(5)$.

We saw that $\Theta(C_5) \ge \frac{1}{2}\log(5)$.

For $x, y \in \mathbf{R}^n$, the *tensor product* of $x$ and $y$ is defined by

$$x\otimes y := (x_1y_1, x_1y_2, \ldots, x_1y_n, x_2y_1, x_2y_2, \ldots, x_2y_n, \ldots, x_ny_{n-1}, x_ny_n).$$

**Observation 5.6**

$$(x \otimes y) \cdot (u \otimes v) = (x \cdot u)(y \cdot v).$$

**Definition 5.7** An *orthonormal representation* of a graph $G$ with $V(G) = \{1, 2, \ldots, n\}$ is a family of unit vectors $u_1, \ldots, u_n$ such that $u_i \cdot u_j = 0$ if $i$ is adjacent to $j$.

**Observation 5.8** If $(g_j)_{j=1}^n$ and $(h_i)_{i=1}^m$ are orthonormal representations of $H$ and $G$, respectively, then $(g_i \otimes h_j)$ is an orthonormal repreesntation of $G \otimes H$.

**Proof.** Given $(i, j) \sim (i', j')$ in $G \otimes H$. Then

$$(g_i \otimes h_j) \cdot (g_{i'} \otimes h_{j'}) = (g_i \cdot g_{i'})(h_j h_{j'}) = 0.$$

$\square$

**Exercise 5.9** If $u_1, \ldots, u_k$ are pairwise orthogonal unit vectors and $c$ is any vector, then

$$\sum_{i=1}^{k} (c \cdot u_i)^2 \leq c^2.$$

(Note that if $c = e_1$ and $\{u_i\}$ is an orthonormal basis, then both sides $= 1$.)

**Lemma 5.10**

$$\omega(G) \leq \theta(G)$$

**Proof.** We may assume that $\{1, \ldots, k\}$ is a max clique in $G$. Then we need that

$$k \leq \max_{1 \leq i \leq n} \frac{1}{(c \cdot u_i)^2}$$

for any orthonormal representation of $G$.

By the exercise, $\sum_{i=1}^{k} (c \cdot u_i)^2 \leq 1$, so $(c \cdot u_i)^2 \leq \frac{1}{k}$, for some $i \in \{1, \ldots, k\}$.

Now we have

$$k \leq \frac{1}{(c \cdot u_i)^2} \leq \max_{1 \leq j \leq n} \frac{1}{(c \cdot u_j)^2}.$$

$\square$

**Lemma 5.11**

$$\theta(G)\theta(H) \geq \theta(G \otimes H).$$

**Proof.** Those are all mins. Let's find the values that hit those minimums.

Let $\{g_1 \ldots, g_n\}$ be an orthonormal representation of $G$ and $c$ a vector such that

$$\theta(G) = \max_{1 \leq i \leq n} \frac{1}{(c \cdot u_i)^2}.$$

Ditto for $\{h_1, \ldots, h_m\}$ and $d$ for $H$.

$$\begin{aligned}
\theta(G \otimes H) &\leq \max_{1 \leq i \leq n, 1 \leq j \leq m} \frac{1}{\left((c \otimes d)(g_i \otimes h_i)\right)^2} \\
&= \max_{i,j} \frac{1}{(cg_i)^2 (dh_j)^2} \\
&= \max_i \frac{1}{(cg_i)^2} \cdot \max_j \frac{1}{(dh_j)^2} \\
&= \theta(G) \cdot \theta(H)
\end{aligned}$$

□

**Corollary 5.12**

$$\omega(G^k) \leq \theta(G^k) \leq \theta^k(G).$$

**Corollary 5.13**

$$\Theta(G) \leq \log(\theta(G)).$$

**Proof.**

$$\begin{aligned}
\Theta(G) &= \lim_{t \to \infty} \log(\omega(G^t)) \\
&= \lim_{t \to \infty} \log(\theta(G)) \\
&= \log(\theta(G))
\end{aligned}$$

**Theorem 5.14**

$$\Theta(C_5) = \frac{1}{2}\log(5).$$

**Proof.** We know $\Theta(C_5) \geq \frac{1}{2}\log(5)$. We need to show that $\theta(C_5) \leq \sqrt{5}$. Hence we need an orthonormal representation and a $c$ such that

$$\frac{1}{(c \cdot u_i)^2} \leq \sqrt{5}.$$

This is a figure, and it's hard to draw. Imagine a 5-pointed umbrella, with $u_1, u_2, u_3, u_4, u_5$ as the spines descending from the top-middle of the open umbrella. The vector $c$ is the middle piece of the umbrella. Arrange the $u$s such that the angle between any 2 non-consecutive spines is 90°. □

**Exercise 5.15**

$$c \cdot u_i = \frac{1}{\sqrt[4]{5}}.$$

The value of $\Theta(C_7)$ is not known.

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