6 A Spectral Characterization of the Lovasz θ function

Recall that a set of unit vectors $u_1, \cdots, u_n$ associated with the vertices $1 \cdots n$ of a graph is called an orthonormal representation of G if $u_i \cdot u_j = 0$ for all $(i,j) \in E$. The Lovasz θ function of a graph is defined as

$$\theta(G) = \min_{c, u_1, \cdots u_n} \max_{1 \le i \le n} \frac{1}{(c \cdot u_i)^2}$$

where c is a unit vector and $u_1, \cdots, u_n$ are an orthonormal representation of G. We shall now see some alternative characterizations of the θ function.

Definition 6.1 Let $c, u_1, \cdots, u_n$ be unit vectors so that the u_i are an orthogonal representation of G, and such that

$$\theta(G) = \max_{1 \le i \le n} \frac{1}{(c \cdot u_i)^2}$$

Then $c, u_1, \cdots, u_n$ is said to be an optimal representation of G.

Fix an optimal representation of G. Let

$$\begin{aligned} z_i &= c - \frac{u_i}{c \cdot u_i} \\ z_i \cdot z_j &= c^2 + \frac{u_i \cdot u_j}{(c \cdot u_i)(c \cdot u_j)} - 2 \\ &= \frac{u_i \cdot u_j}{(c \cdot u_i)(c \cdot u_j)} - 1 \end{aligned}$$

If $(i,j) \in E$, $u_i \cdot u_j = 0$, hence $z_i \cdot z_j = -1$. Since the u_is are unit vectors, $u_i \cdot u_i = 1$, hence

$$z_i \cdot z_i = \frac{1}{(c \cdot u_i)^2} - 1$$

We now define $n \times n$ matrices D, Z and A. Let $z_{ij} = z_i \cdot z_j$. Let

$$d_{ij} = \begin{cases} \frac{1}{(c \cdot u_i)^2} & i = j \\ 0 & i \neq j \end{cases}$$

Finally let $D - Z = A$. It follows that

$$\begin{aligned} a_{ii} &= \frac{1}{(c \cdot u_i)^2} - \left(\frac{1}{(c \cdot u_i)^2} - 1 \right) \\ &= 1 \end{aligned}$$

When $(i, j) \in E$, $a_{ij} = 0 - (-1) = 1$.

We need some definitions and facts from linear algebra.

Definition 6.2 An $n \times n$ real symmetric matrix A is said to be positive semi-definite if $x^T A x \geq 0 \; \forall x \in \mathbf{R}^n$.

Lemma 6.3 *Let A be an $n \times n$ real symmetric matrix.*

- *A has n real eigenvalues and n linearly independent eigenvectors.*
- *A is positive semi-definite iff all its eigenvalues are non-negative.*
- *A is positive semidefinite iff there exists vectors $a_1, \cdots, a_n$ in $\mathbf{R}^n$ such that $a_{ij} = a_i \cdot a_j$.*

Lemma 6.4

- *Let D be an $n \times n$ diagonal matrix. The eigenvalues of D are d_{ii} $1 \leq i \leq n$.*
- *Let $D - A$ be a positive semi-definite matrix. The eigenvalues of A are all smaller than the largest eigenvalue of D.*

We shall use $\lambda_1(A)$ to denote the largest eigenvalue of A. Since $z_{ij} = z_i \cdot z_j$, the matrix Z is positive semi-definite. Further $D - A = Z$, so the eigenvalues of A are all smaller than $\lambda_1(D) = \max_{1 \leq i \leq n} \frac{1}{(c \cdot u_i)^2} = \theta(G)$.

Theorem 6.5

$$\theta(G) = \min_A \lambda_1(A)$$

over all $n \times n$ real symmetric matrices A such that $a_{ij} = 1$ if $i = j$ or $(i, j) \in E$.

Proof. We have constructed $A = D - Z$ which satisfies the above condition such that $\lambda_1(A) \leq \theta(G)$. We shall now show that $\theta(G) \leq \lambda_1(A)$ for all such matrices A. Let A be a matrix such that $\lambda_1(A)$ is minimized and let λ denote this minimum value. Set $B = \lambda I - A$. The eigenvalues of B are $\lambda - \lambda_i(A)$ where $\lambda_i(A)$ denote the eigenvalues of A. Hence the eigenvalues of B are all non-negative and B is a real symmetric matrix, hence it is positive semi-definite. Hence there must exist vectors $x_1, \cdots, x_n$ such that $b_{ij} = x_i \cdot x_j$. We now define a representation of G using these vectors. Let $x_i = (x_{i1}, \cdots, x_{in})$.

$$\begin{aligned}
a_i &= (1, x_{i1}, \cdots x_{in} \quad 1 \leq i \leq n \\
c_i &= (1, 0, \cdots, 0) \\
a_i \cdot a_i &= 1 + \sum_{k=1}^{n} x_{ik}^2 \\
&= 1 + x_i \cdot x_i \\
&= 1 + b_{ii} \\
&= 1 + \lambda - 1 \\
&= \lambda \\
a_i \cdot a_j &= 1 + \sum_{k=1}^{n} x_{ik} \cdot x_{jk} \\
&= 1 + x_i \cdot x_j \\
&= 0 \quad \text{if(i, j)} \in \text{E}
\end{aligned}$$

Hence setting $u_i = \frac{a_i}{\sqrt{\lambda}}$ gives us an orthogonal representation, with $(c \cdot u_i)^2 = \frac{1}{\lambda}\ \forall i$. Hence, it follows that $\theta(G) \leq \lambda$. □

An important corollary of the proof is the existence of optimal orthogonal representations where $c \cdot u_i$ is equal $\forall i$.

Corollary 6.6 *For every graph G, there exists a representation $u_1, \cdots, u_n$ and a vector c such that*

$$\theta(G) = \frac{1}{(c \cdot u_i)^2}\ \forall i = 1, \cdots, n$$

7 A Polyhedral Characterization of the Lovasz θ Function

We now give a polyhedral characterization of the θ function. Recall the following definitions.

Definition 7.1 $STAB(G)$ is the convex hull of the incidence vectors of stable sets in G.

Definition 7.2

$$QSTAB(G) \;=\; \{x \in \mathbf{R}^V, x \geq 0, \sum_{v \in K} x_v \leq 1\}$$

$\forall$ cliques K in G.

Definition 7.3 The theta body $TH(G)$ of a graph is defined as

$$TH(G) = \{x \in \mathbf{R}^V | x \geq 0, \sum_{i=1}^{n} (c \cdot u_i)^2 x_i \leq 1\}$$

for all orthonormal representations $u_1, \cdots, u_n$ and vectors c.

We had seen that $STAB(G)$ is contained in $QSTAB(G)$. We will now show a strengthening of this theorem.

Theorem 7.4

$$STAB(G) \subset \; TH(\overline{G}) \subset \; QSTAB(G)$$

Proof. We shall show that

$$STAB(\overline{G}) \subset \; TH(G) \subset \; QSTAB(\overline{G})$$

$STAB(\overline{G})$ is the convex hull of incidence vectors of the cliques in G. We will show that for every clique K

$$\sum_{i \in K} (c \cdot u_i)^2 \leq 1$$

This follows from the proof that $\omega(G) \leq \theta(G)$ using the fact that the u_is for vertices in a clique are mutually perpendicular. Since every vertex of the polytope $STAB(\overline{G})$ lies in $TH(G)$, the entire polytope must lie in $TH(G)$.

We now show that $TH(G) \subset QSTAB(\overline{G})$. Consider an independent set I in G. Set $u_i = c \; \forall i \in I$ and $u_j \cdot c = 0 \; \forall j \notin I$. The constraint on $TH(G)$ corresponding to this representation is

$$\sum_{i \in I} x_i \;\leq\; 1$$

Hence every point in $TH(G)$ also lies in $QSTAB(\overline{G})$. $\square$

$TH(G)$ is described by an infinite set of inequalities and may not in general be a polytope. When G is perfect, $STAB(G) = TH(\overline{G}) = QSTAB(G)$ and $TH(G)$ is a polytope. This condition is also necessary for $TH(G)$ to be a polytope.

Theorem 7.5 *$TH(G)$ is a polytope iff G is perfect.*

We now give another characterization of the θ function in terms of the theta body of G.

Theorem 7.6

$$\theta(G) = \{max \sum_{i=1}^{n} x_i | x \in TH(G)\}$$

Proof. We shall only show that $\theta(G)$ is an upper bound on the objective function. Consider an optimal representation where $\theta(G) = \frac{1}{(c \cdot u_i)^2}$ $\forall i$.

$$\begin{aligned} \sum_{i=1}^{n} (c \cdot u_i)^2 x_i &\leq 1 \quad \Rightarrow \\ \frac{1}{\theta(G)} \sum_{i=1}^{n} x_i &\leq 1 \quad \Rightarrow \\ \sum_{i=1}^{n} x_i &\leq \theta(G) \end{aligned}$$

$\square$

We now consider optimizing the same objective function $\sum_i x_i$ over $STAB(\overline{G})$ and $QSTAB(\overline{G})$. Clearly the maximum over $STAB$ is attained at a vertex and its value is $\omega(G)$. The optimization problem over $QSTAB$ is

$$\begin{aligned} \max \sum_{i=1}^{n} x_i & \\ x_i \geq 0 & \qquad \forall i \in V \\ \sum_{i \in I} x_i \leq 1 & \qquad \forall I \end{aligned}$$

where I runs over all independent sets in G. If we insist that the x_is be integral, then the optimum value is $\omega(G)$ since if $(i,j) \in E$, then x_i and x_j cannot both be nonzero. Hence this program is a relaxation of the program for clique. We define the fractional clique number $\omega^*(G)$ to be the objective of this function. Clearly $\omega^*(G) > \omega(G)$.

The dual program is

$$\begin{aligned} \min \sum_{i=1}^{n} y_i & \\ y_I \geq 0 & \qquad \forall I \\ \sum_{v \in I} y_I \geq 1 & \qquad \forall V \end{aligned}$$

This is a program to cover all the vertices with the minimum possible number of independent sets. If we add the integrality condition, the optimum equals the chromatic number $\chi(G)$. The optimum of the relaxation is called the fractional chromatic number $\chi^*(G)$. Clearly $\chi^*(G) \leq \chi(G)$.

Corollary 7.7

$$\omega(G) \ \leq \theta(G) \ \leq \omega^*(G) \ \leq \chi^*(G) \ \leq \chi(G)$$

Proof. The first two inequalities come from the inclusion relation $STAB(\overline{G}) \subset TH(G) \subset QSTAB(\overline{G})$. $\omega^*(G) \leq \chi^*(G)$ by LP duality and $\chi^*(G) \leq \chi(G)$ has already been proved. □

Theorem 7.8 *$\theta(G)$ is computable in polynomial time by the ellipsoid method.*

By computable we mean that it can be efficiently approximated to arbitrary precision.

Corollary 7.9 *$\chi(G) = \omega(G)$ can be computed in polynomial time for perfect graphs.*

When G is perfect, $\chi(G) = \omega(G) = \theta(G)$, which can be computed in polynomial time.

References