8 Partitionable Graphs

Theorem 8.1 *A graph G is perfect if and only if $|V(H)| \leq \alpha(H)\omega(H)$ for any induced subgraph H of G.*

Proof. Clearly, $|V(H)| \leq \alpha(H)\chi(H)$ for any induced subgraph H of G. Suppose G is perfect. Then $\chi(H) = \omega(H)$. It follows that $|V(H)| \leq \alpha(H)\omega(H)$.

The "only if part" follows immediately from the next theorem. □

Definition 8.2 A graph G is minimally imperfect if G is not perfect but $G \backslash v$ is perfect for every $v \in V(G)$.

Theorem 8.3 *If G is minimally imperfect, then $|V(G)| = \alpha(G)\omega(G) + 1$.*

Proof. (due to Gaspariyan) Clearly

$$\begin{aligned}
|V(G)| &= |V(G\backslash v)| + 1 \\
&\leq \alpha(G\backslash v)\chi(G\backslash v) + 1 \\
&= \alpha(G\backslash v)\omega(G\backslash v) + 1 \\
&\leq \alpha(G)\omega(G) + 1.
\end{aligned}$$

On the other hand, let $\alpha\colon = \alpha(G)$ and $\omega := \omega(G)$. Let $S_0 = \{v_1, v_2, \ldots, v_\alpha\}$ be a maximum stable set in G. Note that $\omega(G\backslash v) = \omega(G)$ because G is minimally imperfect. Let $S_{(i-1)\omega+1}, S_{(i-1)\omega+2}, \ldots, S_{(i-1)\omega+\omega}$ be an ω-coloring of $G\backslash v_i$, where $i = 1, 2, \ldots, \alpha$. This gives $\alpha\omega + 1$ independent sets in G. Note that $\omega(G\backslash S_i) = \omega$. Thus for all $i = 0, 1, \ldots, \alpha\omega$, there exists a maximum clique Q_i disjoint from S_i.

Claim 1: For any clique Q of size ω, $\exists$ unique $i \in \{0, 1, \ldots, \alpha\omega\}$ such that $Q \cap S_i = \emptyset$.

Proof. If $v_i \notin Q$, then Q is an ω-clique in $G\backslash v_i$, thus Q intersects $S_{(i-1)\omega+1}$, $S_{(i-1)\omega+2}, \ldots, S_{(i-1)\omega+\omega}$. If $S_0 \cap Q = \emptyset$, then Q intersects S_i for all i. So we may assume that $v_1 \in Q$. Then Q meets all S_i's except possibly the first $S_1, \ldots, S_\omega$. But $Q\backslash v$ is an $(\omega - 1)$-clique of $G\backslash v$. Hence Q intersects all but one of the sets in $S_1, \ldots, S_\omega$. This proves the claim. □

By Claim 1, $|Q_i \cap S_j| = 1 - \delta_{ij}$. So $\mathbf{1}_{Q_i} \cdot \mathbf{1}_{S_j} = 1 - \delta_{ij}$, where $\mathbf{1}_{Q_i}$, $\mathbf{1}_{S_j}$ are incidence vectors.

Claim 2: $\mathbf{1}_{S_0}, \ldots, \mathbf{1}_{S_{\alpha\omega}}$ are linearly independent.

Proof. Suppose that $\sum_{i=0}^{\alpha\omega} \lambda_i \mathbf{1}_{S_i} = 0$. Then

$$0 = \sum_{i=0}^{\alpha\omega} \lambda_i \mathbf{1}_{S_i} \cdot \mathbf{1}_{Q_j} = \sum_{i=0, i\neq j}^{\alpha\omega} \lambda_i$$

Thus $\lambda_j = \sum_{i=0}^{\alpha\omega} \lambda_i \ \forall\ j = 0, \ldots, \alpha\omega$. Hence $\lambda_0 = \cdots = \lambda_{\alpha\omega}$, and it follows that $\lambda_i = 0$ for all i. □

Since those vectors in Claim 2 are linearly independent, the dimension of the underlying space is at least $\alpha\omega + 1$. It follows that $|V(G)| \geq \alpha\omega + 1$. □

Corollary 8.4 *G is perfect $\Leftrightarrow$ $\overline{G}$ is perfect.*

Definition 8.5 Let $\alpha, \omega \geq 2$ be integers. A graph G is called (α, ω)-*parrtitionable* or *partitionable* if $\forall v \in V(G)$, the graph $G \backslash v$ can be partitioned into ω stable sets of size α and α cliques of size ω.

Theorem 8.6 *(Lovász) If G is minimally imperfect, then G is partitionable.*

Proof. Let $S_0, \ldots, S_{\alpha\omega}$ be as before. Then $S_1, \ldots, S_\omega$ give a desired partition for $G \backslash v_1$. But every vertex is in a maximum stable set, and so we may choose v_1 to be any vertex we want. By applying the same argument to $\overline{G}$, we get partitions into cliques. □

Observation 8.7 If G is (α, ω)-partitionable, then
(i) $\overline{G}$ is (ω, α)-partitionable.
(ii) $\alpha(G) = \alpha$ and $\omega(G) = \omega$.

Proof. (ii): Pick $v \in V(G)$. Then $\omega(G \backslash v) = \omega$. If $\omega(G) > \omega$, then $\omega(G \backslash u) > \omega$ for some vertex u, a contradiction. Thus $\omega(G) = \omega$. □

Observation 8.8 No partitionable graph is perfect.

Theorem 8.9 *Perfection is in Co-NP.*

Proof. If G is not perfect, exhibit a partitionable subgraph. $\square$

A small improvement: say a graph H is a *small obstacle* if $\exists$ integers $\alpha, \omega \geq 2$ and vertices $a, b, x, y \in V(H)$ such that
(i) $|V(H)| = \alpha\omega + 1$
(ii) $ab \in E(H)$, $xy \notin E(H)$
(iii) $H \backslash a$ and $H \backslash b$ can be partitioned into α cliques.
(iv) $H \backslash x$ and $H \backslash y$ can be partitioned into ω stable sets.

Theorem 8.10 *(Sebö) G is perfect $\Leftrightarrow$ no induced subgraph of G is a small obstacle.*

Proof. "$\Leftarrow$": If G is not perfect, then G has a partitionable subgraph H, which is a small obstacle.

"$\Rightarrow$": WMA G is a small obstacle. Let a, b, x, y be as in the definition.

Claim: $\alpha(G) \leq \alpha$.

Proof. Suppose not. Let S be a maximum stable set of G. Then $a, b \in S$ by (iii), contrary to (ii). $\square$

Similarly, $\omega(G) \leq \omega$ and so G is not perfect by (i). $\square$

Exercise 8.11 If G is $(\alpha, 2)$-partitionable, then $G \cong C_{2\alpha+1}$.

Theorem 8.12 *Let G be a partitionable graph on n vertices. Then*
(i) G has exactly n maximum cliques
(ii) G has exactly n maximum stable sets
(iii) $\forall v \in V(G)$ is in exactly ω maximum stable sets
(iii) $\forall v \in V(G)$ is in exactly α maximum cliques.

Proof. It suffices to prove (i). (iii) follows by examining the stable sets from Gasparian's proof, and (ii) and (iv) follow by taking the complement. Let $n = \alpha\omega + 1$. Let $S_0, \ldots, S_{\alpha\omega}$ and $Q_0, \ldots, Q_{\alpha\omega}$ be as in Gaspariyan's proof. Will prove that $Q_0, \ldots, Q_{\alpha\omega}$ are all the maximum cliques in G. Let A be the matrix with rows $\mathbf{1}_{S_0}, \ldots, \mathbf{1}_{S_{\alpha\omega}}$. Gaspariyan's proof tells us that A is nonsingular. Let $\mathbf{j}_i$ be the column vector whose entries are all 1's except the i^{th} position, which is 0. For any maximum clique Q, $A\mathbf{1}_Q = \mathbf{j}_i$ for some i, by Claim 1 in Gaspariyan's proof. Thus $A(\mathbf{1}_Q - \mathbf{1}_{Q_i}) = \mathbf{j}_i - \mathbf{j}_i = 0$, and hence $Q = Q_i$. $\square$