**Theorem 8.13** *Let $G$ be $(\alpha, \omega)$- partitionable and let $v \in V(G)$. Then $G \backslash v$ is uniquely colorable.*

**Proof.** An $\omega$-coloring of $G \backslash v$ gives a solution to

$$Ax = \mathbf{1} - e_i$$

where, $v$ corresponds to the $i^{th}$ column of $A$ and $e_i = (0 \ldots 1 \ldots 0)^T$, a $|V(G)| \times 1$ vector with the $i^{th}$ entry 1 and the rest zero. Since $A$ is non-singular, the solution is unique. □

**Corollary 8.14** *If $G$ is minimal imperfect and $v \in V(G)$, then $G \backslash v$ is a uniquely colorable perfect graph.*

**Definition 8.15** Let G be a graph with $\omega(G) = \omega$. A clique-pair in $G$ is a pair of cliques $(Q_1, Q_2)$, each of size $\omega$, such that $|Q_1 \bigcap Q_2| = \omega - 1$.

Note: In a $\omega$-coloring the vertices in $Q_1 - Q_2$ and $Q_2 - Q_1$ must recieve the same color.

**Conjecture 8.16** *Every uniquely colorable perfect graphs has a clique pair.*

The above result has been proved for perfect graphs $G$ with $\omega(G) \leq 3$ by Fonlupt and Zemirline. But it is open in general.

**Observation 8.17** Let $(Q_1, Q_2)$ be a clique pair in $G$, let $\omega = \omega(G)$ and let $\{v_1\} = Q_1 - Q_2$ and $\{v_2\} = Q_2 - Q_1$. Let $G'$ be obtained from $G$ by identifying $v_1$ and $v_2$. Then $G$ is uniquely $\omega$-colorable if and only if so is $G'$.

**Conjecture 8.18** *For every uniquely colorable perfect graph there exists a sequence of identifications as above that eventually produces $K_\omega$ (not necessarily staying perfect).*

Note: If true, this gives a certificate for unique colorability. (NP-Hard to decide unique colorability for non-perfect graphs)

**Theorem 8.19** *Every partitionable graph $G$ satisfies*

1. *$\forall u \neq v \in V(G), \exists$ max clique containing $u$ and not $v$.*

2. *If $S_1$,$S_2$ are maximum stable sets, then $S_1 \triangle S_2$ induces a connected graph.*

3. *If $v_1, v_2 \in V(G)$, distinct, then $\exists$ odd induced $v_1, v_2$-path in $G$.*

**Proof.**

1. Immediate from the construction of $S_0, \dots S_{\alpha\omega}$

2. Proof by contradiction. Let $S_1 \triangle S_2$ be disconnected and partition $S_1$ and $S_2$ as follows

$$S_1 = S_{11} \bigcup S_{12};\ S_2 = S_{21} \bigcup S_{22}$$

such that $S_{11} \bigcap S_{21} = S_1 \bigcap S_2$ and there is no path from $S_{12} \bigcup S_{22}$ to $S_{11} \bigcup S_{21}$. Let $S'_2 = S_{21} \bigcup S_{12}$ and $S'_1 = S_{11} \bigcup S_{22}$.
Let $Q_1$ be the unique clique disjoint from $S_1$. Then $Q_1 \bigcap S_2 \neq \emptyset$. Let $Q_1 \bigcap S_2 = \{v\}$. We may assume that $v \in S'_2 - S'_1$. But then $Q_1 \bigcap S'_1 = \emptyset$, a contradiction.

3. Choose $S_1$ and $S_2$ such that $\{v_1, v_2\} \bigcap S_1 = \{v_1\}$ and $\{v_1, v_2\} \bigcap S_2 = \{v_2\}$. Since $S_1 \triangle S_2$ is connected, find an induced $v_1, v_2$-path in $S_1 \triangle S_2$. That path is odd.

$\square$

Let G ve $(\alpha, \omega)$-partionable, and let $S_0, S_1, \dots S_{\alpha\omega}$, $Q_0, \dots, Q_{\alpha\omega}$ be the maximum stable sets and maximum cliques such that $S_i \bigcap Q_i = \emptyset$(Then $S_i \bigcap Q_j \neq \emptyset, i \neq j$).
Reminder: $A$ is the matrix with rows $\mathbf{1}_{S_0}, \mathbf{1}_{S_1}, \dots, \mathbf{1}_{S_{\alpha\omega}}$. Let $B$ be the matrix with rows $\mathbf{1}_{Q_0}, \dots, \mathbf{1}_{Q_{\alpha\omega}}$. Then,

$$A.B^T = J - I = \begin{pmatrix} 0 & & \\ \mathbf{1} & \ddots & \mathbf{1} \\ & & 0 \end{pmatrix}$$

**Lemma 8.20** *(Bridges, Ryser) Let $A, B$ be $(n+1) \times (n+1)$ 0,1-matrices such that $AB^T = J - I$. Then, $AB^T = B^T A$.*

**Proof.** $(J-I)^{-1} = \frac{1}{n}J - I$. To see this,

$$(\frac{1}{n}J - I)(J-I) = \frac{1}{n}J.J - J - \frac{1}{n}J + I = I$$

Therefore,

$$I = A.B^T(J-I)^{-1} = A.B^T(\frac{1}{n}J - I)$$

$$I = A^{-1}A = A^{-1}AB^T(\frac{1}{n}J - I)A = B^T(\frac{1}{n}J - I)A = \frac{1}{n}B^TJA - B^TA$$

$$B^TA = \frac{1}{n}B^TJA - I = \frac{1}{n}(B^T.\mathbf{1})(A^T.\mathbf{1}) - I$$

$0 = tr(AB^T) = tr(B^TA)$, but, $B^TA$ has postive entries and hence $B^TA$ has zeroes on the diagonal. If $B^T.\mathbf{1} = \begin{pmatrix} r_0 \\ \vdots \\ r_n \end{pmatrix}$ and $A^T.\mathbf{1} = \begin{pmatrix} s_0 \\ \vdots \\ s_n \end{pmatrix}$, $r_is_i = n$.

$B^TA$ is integral $\Rightarrow n|r_is_j \Rightarrow \frac{r_i}{s_j}n$ is an integer. Or, as $n = r_is_i, \forall i$, $\frac{r_i}{r_j}$ and $\frac{s_j}{s_i}$ are integers $\forall i, j$.

$$\Rightarrow s_0 = s_1 = ... = s_n$$

$$\Rightarrow r_0 = r_1 = ... = r_n$$

$$\Rightarrow \frac{1}{n}(B^T.\mathbf{1})(A^T.\mathbf{1}) = J$$

$$\Rightarrow B^TA = J - I$$

□

**Lemma 8.21** $Q_i \cap Q_j \neq \emptyset \Rightarrow S_i \cap S_j = \emptyset$

**Proof.** Suppose $v_l \in Q_i \bigcap Q_j$. As $J - I = B^TA, \forall m$, $1 = (B^TA)_{lm}$= number of indices $k$ such that $v_l \in Q_k$ and $v_m \in S_k$. Since, $v_l \in Q_i \bigcap Q_j$, and entries in $A, B$ are positive, $v_m$ is in either $S_i$ or $S_j$ but not both. Or, $S_i \bigcap S_j = \emptyset$. □

**Theorem 8.22** *Let $G$ be $(\alpha, \omega)$ partitionable, let $S_0, ...S_{\alpha\omega}$, $Q_0, ..., Q_{\alpha\omega}$ be all the maximum stable sets and all the maximum cliques respectively such that $S_i \cap Q_i = \emptyset$ (and hence $S_i \cap Q_j \neq \emptyset$ for $i \neq j$). If $S_{i_1}, S_{i_2}, ...S_{i_\omega}$ is the unique $\omega$-coloring of $G \backslash v$, then $Q_{i_1}, ..., Q_{i_\omega}$ are precisely the maximum cliques containing $v$.*