Theorem 8.23 *Let G be (α, ω) partitionable. Let $S_0, S_1, ..., S_{\alpha\omega}$ be all the maximum stable sets. Let $Q_0, Q_1, ..., Q_{\alpha\omega}$ be all the maximum cliques numbered so that $S_i \cup Q_i = \phi$ $\forall i$. Then for $v \in V(G)$, if $S_{i_1}, ..., S_{i_\omega}$ is the unique coloring of $G \backslash v$, then $Q_{i_1}, ..., Q_{i_\omega}$ are precisely the cliques containing v.*

Proof. Let $Q_{i_1}, ..., Q_{i_\omega}$ be the cliques containing v. Then $S_{i_1}, ..., S_{i_\omega}$ are pairwise disjoint, and hence they form an ω-coloring of $G \backslash v$. But that coloring is unique. □

9 Stable Matchings

Bipartite graph B with bipartition (Me, Wo). A marriage assignment is a matching. Every $v \in V(B)$ has a linear order on its neighbors. A matching M is unstable if $\exists$ an edge $e = u_1u_2 \in E(B) - M$ s.t. $\forall i \in \{1, 2\}$ either u_i is not matched or is matched with x, but he prefers u_{3-i} to x.

A matching is stable if it is not unstable.

Theorem 9.1 *(Gale, Shapely) For every bipartite graph B and every assignment of preferences there is a stable matching.*

Proof. In round $2i+1$ every man proposes to the woman he likes the most that hasn't rejected him yet. In round $2i+2$ every woman rejects every suitor except her best suitor. After at most $2|E(B)|$ rounds there will be no more rejections. At that moment every woman marries her only suitor, if she has one.
Claim: This is a stable marriage.
Proof: Pick $e = mw \in E(B) - M$.
Case 1: m proposed to w at some point but got rejected at some round $\Rightarrow$ w married someone else she liked better than m.
Case 2: m never proposed to $w \Rightarrow m$ only proposed to women he liked better $\Rightarrow$ he married a woman he liked better. □

Let $G = L(B)$. If $e, f \in V(G) = E(B)$ are adjacent (say both incident with $v \in V(B)$), then direct the edge $ef \in E(G)$ from e to f if v prefers the other end of e to the other end of f. This gives an orientation D of G.

A stable matching in B is a set $S \subset V(D)$ s.t.

1. S is a stable set, and

2. $\forall v \in V(D) - S,\ \exists s \in S$ s.t. $\overrightarrow{sv} \in E(D)$

A set satisfying 1 and 2 is called a kernel in D.

The theorem of Gale and Shapely can be restated as:

If G is a line graph of a bipartite graph and D is an orientation of G acyclic on every clique, then D has a kernel.

Question: For what graphs is this true?

Example: It is not true for odd holes. This is easy to verify.

Definitions: A directed graph D is a *super-orientation* of an undirected graph G if D is obtained from G by replacing each edge either by a directed edge or a pair of opposite directed edges with the same ends.

An edge $\overrightarrow{uv} \in E(D)$ is called *irreversible* if $\overrightarrow{vu} \notin E(D)$

A directed graph D is *clique-acyclic* if no clique of D has a hamiltonian cycle of irreversible edges.

Homework: Let G be an odd antihole. Find a clique-acyclic super-orientation of G that has no kernel.

Conjecture 9.2 *(Berge, Duchet) G is perfect $\Leftrightarrow$ every clique-acyclic super-orientation has a kernel.*

Proof. ($\Leftarrow$) follows from strong perfect graph conjecture.
($\Rightarrow$) Proved by Boros and Gurvich. We will show a proof by Aharoni and Holzman.

Let D be a digraph. A function $f : V(D) \to R_0^+$ is called *fractionally independent* if $\sum_{v \in K} f(v) \leq 1\ \forall$ clique K.

$S \subset V(D)$ is *dominating* if $S \cap I(v) \neq \phi\ \forall v \in V(D)$, where $I(v) = \{u \in V(d) : u = v \text{ or } \overrightarrow{uv} \in E(D)\}$.

We say that a function $f : V(D) \to R_0^+$ is *fractionally dominating* if $\sum_{v \in I(u)} f(v) \geq 1\ \forall u \in V(D)$.

f is called a *fractional kernel* if it is fractionally independent and fractionally dominating.

If $f = \mathbf{1}_X$ then f is a fractional kernel iff f is a kernel.

We say f is *strongly fractionally dominating* if $\forall u \in V(D)\ \exists$ a clique $K \subseteq I(u)$ s.t $\sum_{v \in K} f(v) \geq 1$.

We say f is a *strong fractional kernel* if f is fractionally independent and strongly fractionally dominating.

Theorem 9.3 *(Aharoni, Holzman) Every clique acyclic digraph has a strong fractional kernel.*

Corollary 9.4 *(Boros,Gurvich) If G is perfect then every clique-acyclic super-orientation of G has a kernel.*

Proof. Let D be a clique-acyclic super-orientation of G. By Aharoni-Holzman $\exists$ a strongly fractional kernel f. The maximum f-weight of a clique in D is 1. So the replication lemma implies that $\exists$ a stable set S that meets every maximum weight clique.

Claim: S is a kernel. To prove this we must show that S is dominating. Let $u \in V(D)$. Since f is strongly fractionally dominating, $\exists$ a clique $K \subset I(u)$ s.t. $\sum_{v \in K} f(v) \geq 1$
$\Rightarrow K$ is a maximum f-weight clique
$\Rightarrow K \cap S \neq \phi$
$\Rightarrow S \cap I(u) \neq \phi$
$\Rightarrow S$ is dominating $\square$