In order to complete the proof that every clique-acyclic superorientation of a perfect graph has a kernel, we have to prove the Aharoni-Holzman theorem:

Theorem 9.5 [**Aharoni-Holzman**] *Every clique-acyclic digraph has a strong fractional kernel.*

Proof. Let D be a clique-acyclic digraph with vertex set V and edge set E. Let $Q_1, ..., Q_m$ be all the maximal cliques in D. We will construct a new graph D' by adding one new vertex z_i for every clique Q_i and connecting each vertex of Q_i to z_i. Hence $D' = (V', E')$, where $V' = V \cup \{z_1, ..., z_m\}$ and $E' = E \cup \{\vec{uz_i} : u \in Q_i\}$. Let $Q'_i := Q_i \cup \{z_i\}$. Clearly the cliques $Q'_1, ..., Q'_m$ are all the maximal cliques of D'. The graph D' is clique-acyclic since the new vertices that we added are sinks, i.e., no edge is coming out of the z_i's. Therefore there exists a linear order $>_i$ on Q'_i such that if $\vec{uv}$ is an irreversible edge, then $u >_i v$. Notice also that z_i is $>_i$-minimal in Q'_i. We can extend each linear order $>_i$ to $V(D) \cup \{z_i\}$ such that:

$$w >_i v \ \ \forall w \in V(D) - Q'_i, \ \forall v \in Q'_i \tag{4}$$

We can further extend $>_i$ to $V(D')$ such that $z_j >_i v \quad \forall j \neq i, \forall v \in V(D) \cup \{z_i\}$.

Let $w_1 := z_1, w_2 := z_2, ..., w_m := z_m$ and let $w_{m+1}, ..., w_n$, be the vertices of D. Let B be the clique-vertex incidence matrix of D'. The rows of B are indexed by the cliques $Q'_1, ..., Q'_m$ and the columns are indexed by the vertices of D', $w_1, ..., w_n$. Furthermore $B_{ij} = 1$ if vertex w_j is in the clique Q'_i and $B_{ij} = 0$ otherwise. An easy observation that will be used later on is that the m leftmost columns of B form the $m \times m$ identity matrix.

Let $b_1, ..., b_n$ be the columns of B. Let $J \subseteq [n]$ be a subset of the columns of B. For a vector $b \in R^m$, we say that J is a *feasible basis* for (B, b) if the columns $(b_j)_{j \in J}$, when seen as vectors, are linearly independent and $b = \sum_{j \in J} \alpha_j b_j$ for some $\alpha_j \geq 0$. In other words, b is in the cone generated by the columns $b_j, j \in J$.

Let $w \in V(D')$, $J \subseteq [n]$. We say that w is subordinated by J at the clique Q'_i if $w <_i w_j \quad \forall j \in J$. We say that w is subordinated by J if it is subordinated by J at some clique Q'_i. We also say that J is subordinating if every vertex is subordinated by J. Note that if J is subordinating, then every subset of J is also subordinating.

The proof of the theorem is based on the following lemma due to Scarf:

Lemma 9.6 [Scarf] *There exists a set $J \subseteq [n]$ which is both a feasible basis for $(B, \mathbf{1})$ and subordinating.*

We will first show how to complete the proof assuming the lemma. Let J be a feasible basis for $(B, \mathbf{1})$ which is also subordinating, as guaranteed by the lemma. Since J is a feasible basis, there exist nonnegative numbers $\alpha_1, ..., \alpha_n$ such that $\alpha_j = 0$ for $j \notin J$ and:

$$B\alpha^T = \mathbf{1}, \text{ where } \alpha = (\alpha_1, ..., \alpha_n)$$

The equation above implies that:

$$\sum_{j \in J : w_j \in Q'_i} \alpha_j = 1 \quad \forall i \in [m]$$

We can now define a function $f : V(D) \to R_0^+$ by setting $f(w_j) := \alpha_j$ for $j = m+1, ..., n$. We will show that f is a strong fractional kernel for D. We first show that f is fractionally independent. To see this, let Q be a clique in D. The clique Q is obviously contained in some maximal clique of D hence there exists i such that $Q \subseteq Q_i \subseteq Q'_i$. But then we have:

$$1 = \sum_{j \in J : w_j \in Q'_i} \alpha_j \geq \sum_{j : w_j \in Q} \alpha_j = \sum_{v \in Q} f(v)$$

Therefore f is fractionally independent and it remains to show that f is strongly fractionally dominating. Let $w \in V(D)$ be fixed. Since J is subordinating, w is subordinated by J at some clique Q'_i. That is,

$$w <_i w_j \quad \forall j \in J \tag{5}$$

Let $K := \{w_j \in Q'_i : j \in J\}$. We claim that $K \subseteq I(w)$. First note that for $w \in V(D)$, $I(w)$ is the same in D and D' since all the edges that we added are directed towards the vertices $z_1, ..., z_m$. Suppose the claim was not true. Then there exists a vertex w_j such that $w_j \in K, w_j \notin I(w)$. Since $w_j \in K$, $w_j \in Q'_i$ and $j \in J$. Also $w_j \notin I(w)$ implies that $w_j \neq w$ and $\vec{w_j w} \notin E(D')$. By (4) and (5) we have that $w \in Q'_i$. In fact $w \in Q_i$. Q'_i is a clique and $\vec{w_j w} \notin E(D')$. Therefore $\vec{w w_j} \in E(D')$ and it is an irreversible edge. But this implies that $w >_i w_j$, which is a contradiction to (5). Hence $K \subseteq I(w)$. This also implies that $K \subseteq V(D)$ and hence K is a clique of the original graph D. We now have:

$$1 = (B\alpha^T)_i = \sum_{j \in J : w_j \in Q'_i} \alpha_j = \sum_{j : w_j \in K} \alpha_j = \sum_{v \in K} f(v)$$

Therefore f is fractionally independent and strongly fractionally dominating, i.e., f is a strong fractional kernel for D. $\square$