Definition 9.7 A *quasi-order* $\le$ on a set X is a reflexive and transitive relation on X (that is not necessarily anti-symmetric). A *linear* quasi-order on X is a quasi-order $\le$ such that any two elements are comparable, that is, $\forall x, y \in X$, we have $x \le y$ or $y \le x$ (or both).

Notice that the relation $\equiv$, given by $x \equiv y$ if $x \le y$ and $y \le x$, would then be an equivalence relation, and it follows that $\le$ induces a partial order on the equivalence classes of that relation. The partial order induced by a linear quasi-order is, in fact, a linear order. For a quasi-order $\le$, we define $x < y$ if $x \le y$ and $x \ne y$.

Lemma 9.8 (Scarf) *Let m and n be integers with $m < n$, and let B be an $m \times n$ matrix such that $B_{ij} = \delta_{ij}$ for $1 \le i, j \le m$. Let $b \in \mathbf{R}^m$ be non-negative such that $\{x \in \mathbf{R}^n \mid Bx = b,\ x \ge 0\}$ is bounded. Further, let $\le_1, \le_2, \ldots, \le_m$ be linear quasi-orders on $[n]$ such that*

$$i \le_i j \le_i k \qquad \forall j > m \ \ \forall k \in [m] - \{i\}$$

Then there exists a feasible basis J for (B, b) such that J is subordinating.

Recall that $k \in [n]$ is *subordinated* by J at $i \in [m]$ if $k \le_i j\ \forall j \in J$. We say that k is subordinated by J if $\exists i \in [m]$ such that k is subordinated by J at i. Finally, J is subordinating if $\forall k \in [n]$, k is subordinated by J. Notice that if J is subordinating and $J' \subseteq J$, then J' is also subordinating.

Before we prove Scarf's Lemma, we need two intermediate results, the first of which follows from elementary linear programming theory.

Sublemma 9.9 *Let (B, b) be as in Scarf's Lemma, and assume that (B, b) is* non-degenerate *(that is, b cannot be expressed as a non-negative linear combination of $< m$ columns of B.) If J is a feasible basis for (B, b) and $k \in [n] - J$, then there exists a unique index $j \in J$ such that $(J - \{j\}) \cup \{k\}$ is a feasible basis.*

Sublemma 9.10 *Suppose that $\le_1, \ldots, \le_m$ as given in Scarf's Lemma are* linear *orders, and let J be a subordinating set of size $m - 1$. Then there exist precisely two indices $j \in [n] - J$ such that $J \cup \{j\}$ is subordinating, unless $J \subseteq [m]$, in which case there exists precisely one such index.*

Proof Let J be subordinating with $|J| = m - 1$. For $i \in [m]$, let $t(i)$ be the $\le_i$-least element of J, that is, $t(i) \in J$ and $t(i) \le_i j\ \forall j \in J$. Since J is subordinating, $\forall j_0 \in J\ \exists i \in [m]$ such that $j_0 \le_i j\ \forall j \in J$, that is, j_0 is the $\le_i$-least element of J. In other words, $t : [m] \to J$ is onto. Hence there exist exactly two indices $i_1, i_2 \in [m]$ such that $t(i_1) = t(i_2)$. By symmetry, we may assume that $i_1 = 1$ and $i_2 = 2$. Now, for $p = 1, 2$, define

$$Z_p = \{j \in [n] - J \mid j \text{ is not subordinated by } J \text{ at any } i \in [m] - \{p\}\}$$

Claim: For any $j \in [n] - J$, $J \cup \{j\}$ is subordinating iff $\exists p \in \{1, 2\}$ such that $j \in Z_p$ and j is the $\leq_p$-maximal element of Z_p.

Suppose $J \cup \{j\}$ is subordinating. If j is subordinated by J at some $i_0 \neq 1, 2$, then $t(i_0)$ cannot be subordinated by $J \cup \{j\}$ at i_0 (since $j <_{i_0} t(i_0)$) or at any $i \in [m] - \{i_0\}$ (since $t(i) <_i t(i_0)$). This contradicts the fact that $J \cup \{j\}$ is subordinating, hence j cannot be subordinated by J at any $i_0 \neq 1, 2$. If j is subordinated by J at both 1 and 2, then $t(1)(= t(2))$ cannot be subordinated by $J \cup \{j\}$. It follows that $j \in Z_p$ for some $p \in \{1, 2\}$. In fact, if j is not the $\leq_p$-maximal element of Z_p, then $\exists h \in [n] - J$ such that $h \in Z_p$ and $j <_p h$. But then, h is not subordinated by $J \cup \{j\}$ at p or at any $i \in [m] - \{p\}$ (since $h \in Z_p$). Hence j must be the $\leq_p$-maximal element of Z_p.

Conversely, suppose $\exists p \in \{1, 2\}$ such that $j \in Z_p$ and j is the $\leq_p$-maximal element of Z_p, and let $k \in [n]$. Now if k is subordinated by J at any i other than p, then it is also subordinated by $J \cup \{j\}$ at i since $t(i) <_i j$. If not, then $k \in Z_p$ and k is subordinated by J at p. Also, since j is the $\leq_p$-maximal element of Z_p, $k \leq_p j$, and hence k is subordinated by $J \cup \{j\}$ at p. This means that $J \cup \{j\}$ is subordinating, and thus proves the claim.

The claim immediately implies that there are at most two indices $j \in [n] - J$ such that $J \cup \{j\}$ is subordinating. Note that Z_1 and Z_2 are disjoint, otherwise an element $k \in Z_1 \cap Z_2$ is not subordinated by J. For any $i \in [m]$, we have

$$i \leq_i j \leq_i k \qquad \forall j > m \;\; \forall k \in [m] - \{i\}$$

Hence, if $i \in J$, then $t(i) = i$, since i is the $\leq_i$-minimal element of $[n]$. This means that $\{1, 2\} \not\subseteq J$, since $t(1) = t(2)$.

Now suppose $J \not\subseteq [m]$. Then for any $i \in [m]$, if $i \notin J$, then $t(i) \notin [m]$; otherwise pick $j \in J - [m]$, and we have $j <_i t(i)$, which contradicts the definition of $t(i)$. In other words, for $i \in [m]$, either $t(i) = i$ or $t(i) \notin [m]$. In particular, $J \cap \{1, 2\} = \emptyset$. Now, for $p = 1, 2$

$$t(i) <_i p \qquad \forall i \in [m] - \{p\}$$

Hence p is not subordinated by J at any $i \in [m] - \{p\}$, that is, $p \in Z_p$. Thus, both Z_1 and Z_2 are non-empty, and there are exactly two indices $j \in [n] - J$ such that $J \cup \{j\}$ is subordinating.

Finally, suppose $J \subseteq [m]$. Since $\{1,2\} \not\subseteq J$ and $|J| = m-1$, we get $J = [m] - \{p\}$ for some $p \in \{1,2\}$, say $p = 1$. Then, as before, $1 \in Z_1$. Moreover, for any $j > n$,

$$t(i) = i <_i j \qquad \forall i \in [m] - \{1\}$$

Hence $Z_1 = [n] - J$, and hence $Z_2 = \emptyset$. It follows that there is exactly one index $j \in [n] - J$ such that $J \cup \{j\}$ is subordinating. This proves Sublemma 9.10. $\square$

Now we are ready to prove the main lemma.

Proof (of Scarf's Lemma) For each $\leq_i$ (which is a linear quasi-order), look at the equivalence relation $\equiv_i$ given by $j \equiv_i k$ if $j \leq_i k$ and $k \leq_i j$. Arbitrarily order each of the equivalence classes of $\equiv_i$ linearly, to obtain a linear order $<'_i$. Since $j <'_i k$ implies $j <_i k$ for all j, k, a set that is subordinating with respect to $\leq_1, \ldots, \leq_m$ is also subordinating with respect to $\leq'_1, \ldots, \leq'_m$. Further, by perturbing b slightly, we may assume that (B, b) is non-degenerate, that is, every feasible basis for (B, b) has size exactly m.

Suppose there is no feasible basis for (B, b) that is subordinating with respect to $\leq'_1, \ldots, \leq'_m$. Now define a bipartite graph H with bipartition $(\mathcal{F}, \mathcal{S})$, where $\mathcal{F}$ is the set of feasible bases for (B, b) that contain the index 1, $\mathcal{S}$ consists of all subordinating sets of size m, and $F \in \mathcal{F}$ is adjacent to $S \in \mathcal{S}$ if $F \backslash S = \{1\}$. Note that since the first m columns of B form an identity matrix, and b has non-negative coefficients, $[m]$ is a feasible basis for (B, b), that is, $[m] \in \mathcal{F}$.

Claim 1: If $F \in \mathcal{F}$ has non-zero degree in the above bipartite graph H, then $\deg(F) = 2$, unless $F = [m]$, in which case $\deg(F) = 1$.

Consider $F \in \mathcal{F}$ with non-zero degree, and let $S \in \mathcal{S}$ be adjacent to F. Since F and S both have size m, $|F \cap S| = m-1$, with $F \backslash S = \{1\}$ and $S \backslash F = \{s\}$ for some $s \in [n]$. Since S is subordinating, $F \cap S$ is also subordinating. We shall apply Sublemma 9.10 to $J = (F \cap S)$. If $F = [m]$, then $J = (F \cap S) \subseteq [m]$, and there is exactly one index $j \in [n] - J$ such that $J \cup \{j\}$ is subordinating (in fact, $j = s$). In other words, S is the only neighbor of F and $\deg(F) = 1$. On the other hand, if $F \neq [m]$, then there are precisely two indices $j, j' \in [n] - J$ such that $J \cup \{j\}$ is subordinating. We may assume $j' = s$ and $j \neq s$. Now $j \neq 1$, otherwise F would be subordinating. But then $(J \cup \{j\}) \neq S$ is subordinating, and is hence the only neighbor of F other than S, that is, $\deg(F) = 2$. This proves Claim 1.

Claim 2: If $S \in \mathcal{S}$ has non-zero degree in H, then $\deg(S) = 2$.

Consider $S \in \mathcal{S}$ with non-zero degree, and let $F \in \mathcal{F}$ be adjacent to S.

Again, $|F \cap S| = m - 1$ with $F \backslash S = \{1\}$ and $S \backslash F = \{s\}$ for some $s \in [n]$. Now applying Sublemma 9.9 to $J = F$ and $k = s$, there is precisely one index $j \in F$ such that $F' = (F - \{j\}) \cup \{s\}$ is a feasible basis. Clearly, $j \neq 1$, otherwise F' $(= S)$ is also subordinating. But then F' is the only neighbor of S other than F, that is, $\deg(S) = 2$. This proves Claim 2.

But now Claims 1 and 2 imply that the degrees of all vertices in H are either 2 or 0, except for the vertex $[m]$, which has degree 1. This is a contradiction, and hence it proves Scarf's Lemma. $\square$