10 A lemma about odd paths

A graph G is *Berge* if it has no odd hole and no odd antihole. $X \subseteq V(G)$ is anticonnected if $\overline{G}[X]$ is connected. $A \subseteq V(G)$ is *complete* (*anticomplete*, resp.) to B if every pair $a \in A, b \in B$ of vertices of A and B are adjacent in G(in $\overline{G}$, resp.)

Lemma 10.1 *(Roussel and Rubio) Let G be a Berge graph and let $X \subseteq V(G)$ be anticonnected. Let P be an odd induced path in $G \setminus X$ and let the ends of P be complete to X. Then*

(i) P has an edge with both ends X-complete, or

(ii) there exist $x, y \in X$ and an induced $x - y$ path with the same interior as P, or

(iii) P has length three and there exists an odd induced antipath with ends the internal vertices of P and interior in X.

In the proof we will need a lemma about subsets of a path. Let P be a path, and let $A_1, A_2, \ldots, A_k \subseteq V(P)$. An *i-path* is a maximal subpath of P with at least one internal vertex and with no internal vertex in A_i. An *i,j-path* is a subpath of P with one end in $A_i - A_j$, the other in $A_j - A_i$ and no internal vertex in $A_i \cup A_j$.

Lemma 10.2 *Let P be an odd induced path of length ≥ 3, and $A_1, ..., A_k \subseteq V(P)$. Assume that*

(i) every i-path is even for all $i \in \{1, ..., k\}$,

(ii) every i,j-path has even length for all $i \neq j \in \{1, ..., k\}$.

Then some internal vertex of P belongs to $A_1 \cap ... \cap A_k$.

Proof. Assume first that $k = 2$. Suppose for a contradiction that no internal vertex belongs to $A_1 \cap A_2$. An *i-interval*, $i = 1, 2$, is a maximal subpath of P with both ends in A_i and no vertex in A_{3-i}. Let $P[x_1, y_1], P[x_2, y_2], \ldots, P[x_k, y_k]$ be all the 1- and 2-intervals in P, and let x_0 and y_{k+1} be the ends of P listed such that $x_0, x_1, y_1, x_2, y_2, \ldots, x_k, y_k, y_{k+1}$ occur on P in the order listed. We may assume that $P[x_1, y_1]$ is a 1-interval; then $P[x_2, y_2]$ is a 2-interval,

$P[x_3, y_3]$ is a 1-interval, and so on. We have $k \geq 2$, for otherwise P is an odd 2-path, contrary to (i). Now $P[x_0, x_2]$ is even by (i) and $P[y_1, x_2]$ is even by (ii). Thus $P[x_0, y_1]$ is even, and similarly $P[x_k, y_{k+1}]$ is even. For $j = 2, 3, \ldots, k-1$ the path $P[y_{k-1}, x_{k+1}]$ is even by (i) and the paths $P[y_{j-1}, x_j]$ and $P[y_j, x_{j+1}]$ are even by (ii). Thus $P[x_j, y_j]$ is even. Since $P[x_0, y_1]$, $P[y_1, x_2]$, $P[x_2, y_2], \ldots$ are all even we see that P is even, a contradiction. This proves the lemma when $k = 2$.

If $k \geq 3$, we will proceed by induction. Let $A = A_{k-1} \cap A_k$. It follows from the case $k = 2$ that the sets $A_1, ..., A_{k-2}, A$ satisfy the hypothesis of the lemma, and hence the conclusion follows by induction on k. $\square$

Proof of Lemma 10.1 If P has length one, then (i) holds. Assume now that P has length three, and let x, y be its internal vertices. If one of x, y is complete to X, then (i) holds, and so we may assume that neither of them is. Since X is anticonnected there exists an antipath with ends x, y and interior in X. Let Q be a shortest such antipath. Then Q is induced, and it is odd, for otherwise it could be completed to an odd antihole by adding the ends of P. Thus (iii) holds.

We may therefore assume that P has length at least five. We may assume that X is not stable, for otherwise the lemma follows from Lemma 10.2. We may also assume that no internal vertices of P is X-complete, for otherwise we apply induction on the length of P. We may assume:

($*$) for any anticonnected set $X' \subsetneqq X$ and for every odd subpath P' of P with both ends X'-complete, P' has an edge with both ends X'-complete.

There exist $x_1 \neq x_2 \in X$ such that $X - \{x_1\}, X - \{x_2\}$, $X - \{x_1, x_2\}$ are all anticonnected. Let A_i be the set of all vertices of P that are $X - \{x_i\}$-complete.

Since X is anticonnected there exists and an induced antipath in X with ends x_1 and x_2. Assume first that Q is even. By ($*$), applied to $X - \{x_i\}$ and P, there exists an edge e_i of P with both ends in A_i. Since P has length at least five, P has nonadjacent internal vertices $a_1 \in A_1$ and $a_2 \in A_2$. Now Q can be completed to an odd antihole by adding a_1 and a_2, a contradiction.

Thus we may assume that Q is odd. We apply Lemma 10.2 to the path P and sets A_1 and A_2. Since no internal vertex of P belongs to $A_1 \cap A_2$, and for $i \in \{1, 2\}$ every i-path is even by ($*$), we deduce that there exists an odd $1, 2$-path, say R. If R has length at least three, then by ($*$) applied to $X - \{x_1, x_2\}$ and R there is an $X - \{x_1, x_2\}$-complete vertex r in the interior of R. Now Q can be completed to an odd antihole by adding r, a contradiction. Thus R has length one. Since P has length at least five, at least one end of P, say p, is not adjacent to either end of R. Now Q can be completed to an odd antihole by adding p and the ends of R. $\square$