11 Application of Lemma 10.1

We will start with an application of Lemma 10.1.

Theorem 11.1 *Let G be a Berge graph. Let C be a hole, and let D be an antihole, both of length at least 8. Then $|V(C) \cap V(D)| \leq 3$.*

Proof. Assume $|V(C) \cap V(D)| \geq 4$. In any graph (not necessarily Berge), $|V(C) \cap V(D)| \leq 5$: If there were at least 6 vertices in the intersection, then each such vertex has either degree or antidegree at least 3 in $G[V(C) \cap V(D)]$, which is a contradiction since C and D are (anti)holes. If $|V(C) \cap V(D)| = 5$, then C and D are isomorphic to C_5.
So WMA $|V(C)| \cap |V(D)| = 4$. Let $V(C) = \{p_1, p_2, ..., p_m\}$, and $V(D) = \{q_1, q_2, ..., q_n\}$, where $p_i \sim p_{i+1}(mod m)$, and $q_i \sim q_{i+1}(mod n)$ in G^c. Also let $p_1 = q_2$, $p_2 = q_4$, $p_3 = q_1$, $p_4 = q_3$ be the vertices of the intersection. Let P be the induced path $p_4p_5...p_mp_1$. Then P is odd. Let X be the vertex-set of the antipath $q_5q_6...q_n$. Since p_1, p_4 are nonneighbors and in $G[V(C) \cap V(D)]$, both are complete to X. So by Lemma 10.1, one of the following conditions holds:

(1)$\exists\ p_i, i \in \{5, ..., m\}$ such that p_i is complete to X.

(2)$\exists\ j \in \{5, ..., n-1\}$ and an induced odd path Q between q_j and q_{j+1}, with interior $p_5, p_6, ..., p_m$.

If (1) holds, then since the antipath on the vertex-set X is odd, $p_ip_2Xp_3p_i$ is an odd antihole, a contradiction. Assume (2) holds. WMA q_j is a nonneighbor of p_2 (If not, then q_{j+1} is a nonneighbor of p_3). But then $p_3q_{j+1}p_5p_6...p_mq_jp_3$ would be an odd hole, since this has two more edges than P. □

12 Separations

The main theorem which implies the SPGC says roughly that every Berge graph is either basic, or that it has a separation. *Basic* graphs are bipartite graphs, line graphs of bipartite graphs, complements of those two classes, and some other classes which we won't be concerned with for now.

Definition 12.1 A *2-join* in a graph G is a partition (X_1, X_2) of $V(G)$ such that $\exists$ disjoint, nonempty sets $A_i, B_i \subseteq X_i,\ i \in 1, 2$ satisfying
(1) A_1 is complete to A_2, B_1 is complete to B_2, (2) there are no more edges between X_1 and X_2, and (3) $|X_1|, |X_2| \geq 3$.

Definition 12.2 The *blocks of a 2-join* are defined as follows: If there is no $A_2 - B_2$ path in $G[X_2]$, then pick $a_2 \in A_2$, $b_2 \in B_2$ and define the block G_1 to be $G[X_1 \cup \{a_2, b_2\}]$. Otherwise pick an induced $A_2 - B_2$ path P in $G[X_2]$ with no internal vertex in $A_2 \cup B_2$ and define the block G_1 to be $G[X_1 \cup V(P)]$. The block G_2 is defined analogously.

Theorem 12.3 *Let G be a graph that has a 2-join with blocks G_1, G_2. Then G is perfect iff G_1 and G_2 are perfect.*

Proof. Homework. □

Definition 12.4 A graph G is called *minimally imperfect* if G is not perfect, but $G \setminus v$ is perfect $\forall v \in V(G)$.

Note that the SPGC implies that minimally imperfect graphs are odd holes or odd antiholes. The following implies that a minimal counterexample to the SPGC cannot have a 2-join.

Theorem 12.5 *Let G be a minimally imperfect graph with a 2-join. Then it is an odd hole.*

Proof. Let (X_1, X_2) be a 2-join in G with blocks G_1, G_2. Since G_1 and G_2 are induced subgraphs of G, not both of them can be proper: If there were, then there would exist $v \notin V(G_1)$ and $v \notin V(G_2)$ and by the minimality of G, G_1 and G_2 would be perfect, which would imply the perfection of G by Theorem 12.3, contradiction. So WMA $G = G_1$. Since $|X_2| \geq 3$, there exists an induced path P as in the definition of the block G_1, and an internal vertex v. If $\omega := \omega(G) \geq 3$, then we could use the ω-coloring of the perfect graph $G \setminus v$ to color G, and assign to v a color not used by its two neighbors. This gives an ω-coloring of G, contradiction to G being minimally imperfect. So WMA $\omega = 2$. This means G has an odd cycle, and therefore an induced odd cycle. But this induced odd cycle has length at least 5, since $\omega = 2$. So G has an odd hole, and since G was minimally imperfect, G is an odd hole. □

Definition 12.6 A *minimum imperfect graph* is a graph G that satisfies:
(1) G is Berge
(2) G is not perfect
(3) subject to (1) and (2), $|V(G)|$ is minimum

What we are looking for are "separations" that cannot be contained in any minimum imperfect graph (e.g. 2-joins). Here is another example of such a separation:

Definition 12.7 A *star cutset* in a graph G is a set X s.t. $G \setminus X$ is disconnected, and some $x \in X$ is adjacent to every $v \in X \setminus x$.

Theorem 12.8 *No minimally imperfect graph has a star cutset.*

Proof. Assume it does. Let A_1, A_2 be the union of components of $G \setminus X$, so that $A_1 \cup A_2 \cup X = V(G)$, and $A_1 \cap A_2 = \emptyset$. Let $k := \omega(G)$. For $i \in \{1, 2\}$, $G \setminus A_i$ has a k-coloring. Let I_i be the color class containing x, and let $I = I_1 \cup I_2$. Then I is independent (because X is a star cutset), and $\omega(G \setminus I) \leq k - 1$. By the minimality of G, $G \setminus I$ has a $(k-1)$-coloring. Then adding I back in, we obtain a k-coloring of G, contrary to the imperfection of G. $\square$

In the proof of the SPGC, a generalization of star cutsets is needed.

Definition 12.9 A *skew partition* in a graph G is a partition (A, B) of $V(G)$, s.t. A is not connected, and B is not anticonnected. The components of $G[A]$ are denoted by $A_1, ..., A_k$, and the anticomponents of $G[B]$ are denoted by $B_1, ..., B_l$.

Note that complement of a skew partition is again a skew partition.

Conjecture 12.10 (Chvátal) *No minimally imperfect graph has a skew partition.*

Note that this follows from the SPGC. Instead of proving Chvatál's conjecture directly, we will work with a slight strengthening of skew partitions.

Definition 12.11 A *balanced skew partition* in G is a skew partition s.t. every induced path in G with both ends in B and interior in A is even, and every induced antipath with both ends in A and interior in B is even.

Definition 12.12 A *semi-balanced skew partition* in a graph G is a skew partition (A, B) in G such that there exists an anticomponent B_1 of B, s.t. every induced path in G with both ends in B_1 and interior in A is even, and every induced antipath in G with both ends in A and interior in B_1 is even.

Note that star cutsets are balanced skew partitions. It can be shown that if a Berge graph G has a semi-balanced skew partition, then G has a balanced skew partition. The two conditions for a semi-balanced skew partition can be rephrased as follows for Berge graphs: A Berge graph G has a semi-balanced skew partition iff we can add a new vertex adjacent to every $v \in B_1$, and the graph remains Berge.

Theorem 12.13 *No minimum imperfect graph G has a semi-balanced skew partition.*

Proof. Suppose G is a minimum imperfect graph with a semi-balanced skew partition (A, B). Let B_1 as in the definition of a semi-balanced skew partition and let A_1 be a component of A. WMA $|A_1|, |V(G) \setminus A_1 \setminus B| \geq 2$, because otherwise (B, A) would be a star cutset in G^c. Let $k = \omega(G),\ s = \omega(G[B_1])$. We claim that $\exists X \subseteq V(G)$ s.t. $\omega(G[X]) = s,\ \omega(G \setminus X) \leq k - s,\ X \cap B = B_1$. By the claim, $G[X]$ can be s-colored, $G \setminus X$ can be colored with at most $k - s$ colors, and therefore G is k-colorable, a contradiction. It suffices to prove the claim for the graph $G_1 := G[A_1 \cup B]$, because we can repeat the same argument for $G[A_2 \cup ... \cup A_k \cup B]$, and let the final X be the union of the sets obtained in these two steps.
So let $G_1 = G[A_1 \cup B]$, and redefine $k := \omega(G_1)$. Let $G_1' := G_1 + v$, where v is a new vertex joined to all vertices of B_1. Then G_1' is Berge by the alternative characterization of semi-balanced Berge graphs. Since $|V(G_1')| < |V(G)|$ (B consisted of at least 2 (nonempty) components in G), G_1' is perfect (G is minimum imperfect). Let G_1'' be obtained from G_1 by replicating v $k - s - 1$ times. By the Replication Lemma, G_1'' is perfect. But v and its copies will be part of a clique of size $(k - s - 1) + 1 + s = k$. Since G_1'' is perfect and obviously $\omega(G_1'') = k$, we know that G_1'' has a k-coloring. Now let X be the union of the color classes of the colors used for B_1. Note that we have exactly s such classes, since the copies of v form a clique of size $k - s$, and they are joined to every vertex in B_1. Then $X \cap B = B_1$ (since B_1 is complete to B_i for $i \neq 1$), $\omega(G_1[X]) = s$ (since B_1 has a s-clique) and $\omega(G_1 \setminus X) \leq k - s$ (since $G_1 \setminus X$ can be $(k - s)$-colored). Then X is as desired for G_1. □