Definition 12.21 Let H be a subdivision of a 3-connected graph. We say that $X \subseteq E(H)$ *saturates* $L(H)$ if for every branch-vertex v of H, at most one edge of H incident with v is not in X.

Definition 12.22 Let G be a graph, $L(H)$ be an induced subgraph of G. We say that $v \in V(G) - V(L(H))$ is *major* if the neighbors of v in $L(H)$ saturate $L(H)$.

In the following we will study anticonnected components of major vertices.

Theorem 12.23 *Let J be a 3-connected graph which is not K_4 or $K_{3,3}$. Let H be a bipartite subdivision of J. Let G be a Berge graph such that $L(H)$ is an induced subgraph of G, and let Y be an anticonnected set of major vertices. Then the set of common neighbors of Y in $L(H)$ saturates $L(H)$.*

Proof. Suppose not. Choose Y violating the claim of the theorem with $|Y|$ minimum.

There is a branch vertex $v \in V(H)$ such that at least two edges incident with v are not Y-complete. Since Y is anticonnected there exists an antipath $A = e_1 y_1 \dots y_2 e_2$ where e_1 and e_2 are incident with v and $y_1, y_2 \in Y$. By the minimality of Y, Y *is* an antipath from y_1 to y_2. Since for every vertex in Y, the neighbors of that vertex saturate $L(H)$, $y_1 \neq y_2$ and so $|Y| \geq 2$.

Let X be the set of common neighbors of Y and for $i = 1, 2$ let X_i be the set of common neighbors of $Y - \{y_i\}$ that are not in X. The sets X, X_1 and X_2 are pairwise disjoint and $X \cup X_1$ and $X \cup X_2$ saturate $L(H)$ by the minimality of Y.

Lemma 12.24 *Let H be a bipartite subdivision of a 3-connected graph which is not K_4 or $K_{3,3}$ and let X, X_1, X_2 be disjoint nonempty subsets of $E(H)$ such that*

(a) *for any branch-vertex $b \in V(H)$, either*

- **(a1)** *at most one edge incident with b does not belong to X or*
- **(a2)** *of edges of H incident with b, exactly one belongs to X_1, exectly one belongs to X_2 and all the others belong to X;*

(b) *there does not exist a cycle C in H with edge-set $\{e_1, e_2, e_3, e_4\}$ in order, where $e_1 \in X_1$, $e_2 \in X_2$, $e_3, e_4 \in X$ and the common end of e_1 and e_2 in C is a branch-vertex;*

(c) *if $e_1 \in X_1$ and $e_2 \in X_2$ are adjacent, then everey edge in X is adjacent to e_1 or e_2 or both.*

Then every branch-vertex b satisfies (a1).

If $|Y|$ is even, Lemma 12.24 implies the theorem.

Now we will prove that the conditions are fulfilled, provided that $|Y|$ is even.

Proof of (a). Let b be a branch vertex not satisfying (a1). Since $X \cup X_1$, $X \cup X_2$ saturate $L(H)$, all but one edge in $\delta(b)$ are in $X \cup X_1$ and all but one are in $X \cup X_2$ therefore (a2) holds.

Proof of (b) assuming |Y| is even. Suppose there exists such a cycle. Let e be the third edge incident with the common neighbor of e_1 and e_2. Complete A via $e_2e_4ee_3e_1$ to get an odd antihole.

Proof of (c) assuming |Y| is even. Suppose that there exist $e \in X$ which is not adjacent to either of e_1 and e_2. Complete A via $e \in X$ to get an odd antihole.

Lemma 12.25 *Let H be a bipartite subdivision of a 3-connected graph which is not K_4 or $K_{3,3}$ and let X, X_1, X_2 be disjoint nonempty subsets of $E(H)$ such that*

(a) *for any branch-vertex $b \in V(H)$, either*

- **(a1)** *at most one edge incident with b does not belong to X or*
- **(a2)** *of edges of H incident with b, exactly one belongs to X_1, exectly one belongs to X_2 and all the others belong to X;*

(b) *for every even path P in H of lenght at least 4 with both endedges and no internal edge in X every edge of X is incident with an internal vertex of P;*

(c) *every edge in X_1 is adjacent to every edge in X_2.*

(d) *for any three paths P_1, P_2, P_3 in H with common end, say v, and otherwise vertex-disjoint, such that each P_i includes an edge of X, the following is true: at least two of those three edges of $P_1 \cup P_2 \cup P_3$ that are incident with v are in X;*

Then every branch-vertex b satisfies (a1).

If $|Y|$ is odd, Lemma 12.25 implies the theorem.

Again, we will prove that the conditions of the lemma are fulfilled, provided that $|Y|$ is odd.

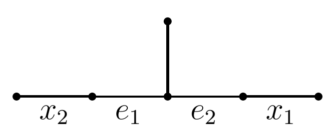
Proof of (a). This condition is exactly the same as condition (a) in Lemma 12.24, and since we did not use the parity of $|Y|$ there, the same proof applies.

Proof of (b) assuming |Y| is odd. Suppose there is an edge $e \in X$ which is not incident with an internal vertex of P. Y is anticonnected, the path in the line graph is odd induced and its ends are complete to Y, so we can apply the Roussel-Rubio Lemma. Outcome (i) contradicts the conditions. In case of outcome (ii), the induced $x - y$ path can be completed via e form an odd hole; in case of outcome (iii) the odd induced antipath can be completed via e to form an odd antihole.

Proof of (c) assuming |Y| is odd. Suppose that there are two non-adjacent edges $e_1 \in X_1$ and $e_2 \in X_2$. Then Y completes to an odd antihole via e_1 and e_2.

Proof of (d) assuming |Y| is odd. Suppose that at most one of the three edges incident with v are in X. We may assume that P_1 and P_2 have the same parity. Then $P_1 \cup P_2$ is even, so by part (b) every edge of X is incident with an internal edge of $P_1 \cup P_2$. So in fact the edge of P_3 which is incident to v is in X.

Denote the edges of X on $P_1 \cup P_2$ by x_1 and x_2. If the number of edges between x_1 and x_2 is more than two, then using condition (a) and the fact that graph is Berge, we can easily conclude that there are non-adjacent edges of X_1 and X_2 on the path, contradicting (c). So this part of the graph must look like the following picture (the thick edges belong to X):



But now we can complete A via $e_1x_1x_2e_2$ to get an odd antihole. □

Lemma 12.24 will be proved in the next lecture. We omit the proof of Lemma 12.25; see [3].