**Theorem 12.26** *Let $H$ be a bipartite subdivision of a 3-connected graph $\neq K_{3,3}$ or $K_4$. Let $X, X_1, X_2 \subseteq E(H)$ be disjoint and non-empty. Assume that*

*(a) For all branch vertices $b \in V(H)$ either*

*(a1) $\leq 1$ edge of $\delta(b)$ is not in $X$*

*(a2) $|\delta(b) \cap X_1| = 1 = |\delta(b) \cap X_2|$ and $|\delta(b) \cap X| = \delta(x) - 2$*

*(b) There does not exist a cycle $C$ in $H$ with edge se $\{e_1, e_2, e_3, e_4\}$ where $e_1 \in X_1, e_2 \in X_2$, and $e_3, e_4 \in X$, and the common end of $e_1$ and $e_2$ in $C$ is a branch vertex.*

*(c) if $e_1 \in X_1, e_2 \in X_2$ are adjacent, then every edge of $X$ is adjacent to $e_1$ or $e_2$ or both.*

*Then every branch vertex satisfies (a1).*

**Proof.**

Suppose not. We select a branch vertex, $b$, and branches $B_1, B_2, B_3$ incident to $b$ such that letting $e_i \in E(B_i)$ be incident to $b$ we have that $e_1 \in X_1, e_2 \in X_2, e_3 \in X$. WMA the pair $(b, B_3)$ is chosen so that $|E(B_3)|$ is maximum. Let us say a branch vertex $z \in V(H)$ is a *triad* if it is incident with $\leq 1$ edge in X. (Note: every triad satisfies (a2) and has degree 3). Let $b_i$ be the vertex at the end of $B_i$.

**Lemma 12.27** *$b_3$ is a triad.*

We will prove this claim by contradiction. Suppose not that $b_3$ is not a triad. Then we can pick $e_1 \in X \backslash E(B_3)$ incident with $b_3$. By (c), this edge must be adjacent to $e_1$ or $e_2$, say $e_1$ (this is WLOG by symmetry). Thus $e = b_3 b_1$ and $E(B_1) = e_1$ and $B_3$ is even.

Pick $e' \in X \backslash \{b_1, b_3\}$ incident with $b_3$. Thus $e' \notin E(B_3)$ by (c) because $B_3$ is even. Using (c) we also see that $e' = b_1 b_2$ and that $E(B_2) = e_2$. This contradicts (b) looking at the cycle $\{e_1, e_2, b_2 b_3, b_3 b_1\}$.

**Lemma 12.28** $|E(B_3)| = 1$

Again, for a contradiction, suppose that $|E(B_3)| \geq 2$. Pick $f_3 \in X$ incident with $b_3$. Then $f_3 \notin E(B_3)$ by c). We also know from c) that $f_3 \sim e_1$ or $f_3 \sim e_2$. By symmetry, we may suppose that $f_3 \sim e_1$.

Pick $f_2 \in X$ incident with $b_2$. Choose it so $f_2 \notin B_2$ if possible. By c) applied to $f_2$, $e_1$, and $e_2$, we see that one of three possibilities arise:

$f = b_1 b_2$ $E(B_2) = f_2, e_2$ $E(B_2) = e_2$

We can see the first two possibilities cannot occur by applying c) to $f_2$ and $b_3$ and observing that our graph can have no odd cycles. Therefore $E(B_2) = e_2$.

Now we claim that $b_3$ and $b$ both have degree 3. Claim 1 showed that $b_3$ is a triad, so it has degree 3. If $b$ is incident with another edge e, this e must be in X by a). However this is contrary to c) applied to e and $b_3$. In this case, we see that H is a subdivision of $K_4$.

**Lemma 12.29** *At least 1 of $B_1, B_2$ has length 1.*

Suppose not, then by c) we get that $b_1, B_2$ are triads and apply c) to $f_2$ and $b_1$ to deduce that $b_1 \sim b_2$. This gives us a pentagon in H, which is a contradiction, since H is bipartite. WMA assume from symmetry that $B_1$ has length 1.

**Lemma 12.30** *$B_2$ has length 1*

Suppose not. Pick $f_2 \in X$ incident to $b_2$. WMA that $f_2 \notin E(B_2)$, for otherwise $(b_2, B_2)$ would violate the choice of (b,B).

So by c) $f_2 = b_1b_2$ and no other edges of X are incident with $b_2$ except possibly $e_2'$. By c) applied to $b_1b_2$ and $b_3$ we see that $b_2b_3 \in E(H)\backslash X$.

If $b_2$ has degree 3 then H is a subdivision of $K_4$ (by the same argument we made in the subclaim of claim 2). WMA $b_2$ has degree $\geq 4$, therefore $e_2' \in X$ and $E(B_2) = e_2', e_2$ by c).

The cycle $e_1, e_2, e_2', b_1b_2$ violates condition b).

Now we investigate the possibility that there is an edge from $b_i$ to an as yet undiscussed vertex $x_i$ For $x = 1, 2$ $f_i = b_ix_i \in X$ then $X_1 \neq X_2$ by b). Then by c) applied to $x_1$ and $b_3$ we have that $x_1 \sim b_3$ and $x_1b_3 \notin X$ and similarly $x_2 \sim b_3$ and $x_2b_3 \notin X$. Now we claim that $b_1$ is a triad. Let $f_1' \in X\backslash\{f_1\}$ be and edge incident with $b_1$. Then $f_1' = b_1x_2$. By c) applied to $f_i$ and $b_3$. However this is contrary to b). We can also see that $b_1 \sim x_2$ by c) applied to $f_2$ and $b_1$, and similarly $b_2 \sim x_1$. We can see also that both $x_1$ and $x_2$ are triads. The graph is forced either to be $K_{3,3}$, or have b as a cut vertex, which is a contradiction.

□