A *skew partition* of a graph $G$ is a partition $(A, B)$ of the vertex set such that $G[A]$ is not connected and $G[B]$ is not anti-connected.

**Definition 12.31** A skew partition $(A, B)$ is *balanced* if every induced path with ends in $B$ and interior vertices in $A$ is even, and every induced anti-path with ends in $A$ and interior vertices in $B$ is even.

**Definition 12.32** Let $(A, B)$ be a skew partition of a graph $G$ and let $A_1, \ldots, A_k$ be the induced connected components of $G[A]$ and let $B_1, \ldots, B_l$ be the induced anti-connected components of $G[B]$. The skew partition is *semi-balanced* if there exists some $i$ such that every induced path with ends in $B_i$ and interior vertices in $A$ is even, and every induced anti-path with ends in $A$ and interior vertices in $B_i$ is even.

We have already seen the following theorem.

**Theorem 12.33** *No minimum imperfect graph has a semi-balanced skew partition.*

A strengthing of the above definition is the idea of weakly balanced.

**Definition 12.34** Let $(A, B)$ be a skew partition where $A_1, \ldots, A_k$ are the induced connected components of $G[A]$ and let $B_1, \ldots, B_l$ are the induced anti-connected components of $G[B]$. The skew partition is *weakly balanced* if there exist an $i$ and a $j$ such that every induced path with ends in $B_i$ and interior vertices in $A_j$ is even, and every induced anti-path with ends in $A_j$ and interior vertices in $B_i$ is even.

**Exercise 12.35** Prove that if a Berge graph has a weakly balanced skew partition, then it has a semi-balanced skew partition.

**Observation 12.36** Let $(A, B)$ be a skew partition in a Berge graph. If some vertex $v \in B$ has no neighbor in a connected component of $G[A]$, then $(A, B)$ is weakly balanced.

**Proof.**

Let $(A, B)$ and $v \in B$ be as above. Without loss of generality, let $B_1$ be the anti-connected component of $B$ containing $v$ and let $A_1$ be a connected component of $A$ with no neighbor of $v$. Let $B_2$ be another anti-connected component of $B$. Then any odd induced anti-path with ends in $A_1$ and internal vertices in $B_2$ can be completed to an odd hole through the vertex $v$, contradicting our assumption that $G$ is Berge. Similarly, an odd anti-path with ends in $A_1$ and all internal vertices in $B_2$ can be completed to an odd anti-hole using the vertex $v$ again contradicting our assumption that $G$ is Berge. □

In the following definitions, let $J$ be a 3-connected graph and $H$ a bipartite subdivision of $J$. Assume that $L(H)$, the line graph of $H$, is an induced subgraph of $H$.

**Definition 12.37** A *$J$-strip system* is a pair $(N, S)$ where $S := \{S_{u,v} : (u, v) \in E(J), S_{u,v} \subseteq V(G)\}$ and $N := \{N_v : v \in V(J), N_v \subseteq V(G)\}$. For $(u, v) \in E(J)$, a *$u, v$-rung* is an induced path $R$ of $G$ with ends $s, t$ where $V(R) \subseteq S_{u,v}$ and $s$ is the unique vertex of $R$ in $N_u$ and $t$ is the unique vertex of $R$ in $N_v$. Moreover, there are the additional properties:

1. $S_{u,v} = S_{v,u}$ are pairwise disjoint.
2. For all $u \in V(J)$, $N_u$ is a subset of $\cup S_{u,v}$ where $v \sim u$ in $J$.
3. For all $(u, v) \in E(J)$, every vertex of $S_{u,v}$ is in a $u, v$-rung.
4. If for $(u, v), (w, x) \in E(J)$ $u, v, w, x$ all distinct, then no edge of $G$ exists between $S_{u,v}$ and $S_{w,x}$.
5. If $(u, v), (u, w) \in E(J)$ where $v \neq w$, then $N_u \cap S_{u,v}$ is complete to $N_u \cap S_{u,v}$ and there are no other $S_{u,v}$ - $S_{u,w}$ edges in $G$.

Then for $u \sim v$ in $J$, define $N_{u,v}$ to be $N_u \cap S_{u,v}$. Then for a fixed $u$ in $V(J)$, $\{N_{u,v} : (u, v) \in E(J)\}$ is a partition of $N_u$.

Notice that $N_{u,v} \neq N_{v,u}$. Also, observe, $L(H)$ gives rise to a $J$-strip system.

We say a $J$-strip system $(S', N')$ extends the $J$-strip system $(S, N)$ if $S_{u,v} \subseteq S'_{u,v}$ and $N_w \subseteq N'_w$ for all $(u, v) \in E(J)$ and $w \in V(J)$.

**Assumption (A)** Let $G$ be a Berge graph and let $J$ be a 3-connected graph, $J \neq K_4, K_{3,3}$. Assume that $J$ is maximal (in number of vertices) such that for some bipartite subdivision $H$ of $J$, $L(H)$ is an induced subgraph of $G$. Let $(S, N)$ be a $J$-strip system extending $L(H)$ with $|V(S, N)|$ maximum where $V(S, N) = \bigcup_{(u,v)\in E(J)} S_{u,v}$.

**Definition 12.38** A vertex $y \in V(G)$ is *major* with respect to a $J$-strip system $(S, N)$ if its neighbors saturate $(S, N)$. In other words, for all $u \in V(J)$ there is at most one neighbor $v$ of $u$ such that $N_{u,v} \subseteq$ the neighbors of $y$.

**Definition 12.39** Let $(S, N)$ be a $J$-strip system in a graph $G$. A set $X \subseteq V(S, N)$ is *local* if either $X \subseteq N_v$ for some vertex $v \in V(J)$ or $X \subseteq S_{u,v}$ for some $(u, v) \in E(J)$.

**Theorem 12.40** *Assume (A) and let $F$ be a connected subset of $V(G) - V(S, N)$ containing no major vertices. Then the set of neighbors of $F$ in $V(S, N)$ is a local set.*

We omit the proof. See [3].

**Theorem 12.41** *Assume (A). Then either $G = L(H)$ or $G$ has a weakly balanced skew partition or $G$ has a 2-join.*

**Proof.** Let $Y$ be the set of major vertices with respect to $(S, N)$

**case 1:** $Y \neq \emptyset$ Let $Y_1$ be an anti-component of $Y$ and let $X$ be the set of common neighbors of $Y_1$. If $X$ does not saturate $(S, N)$, then there is a choice of rungs forming a copy of $L(H)$ such that $X$ does not saturate the copy of $L(H)$. This contradicts an earlier theorem. So we may assume $X$ does saturate the strip system.

Choose $(b_1, b_2) \in E(J)$ such that $S_{b_1,b_2}$ is not contained in $X$ if possible and otherwise choose arbitrarily. Let $X_1$ be defined to be the union of all but one of the sets ($N_{b_1,v}$ for $v \sim b_1$ that are contained in $X$, leaving out $N_{b_1,b_2}$ if possible. Similarly, define $X_2$. Let $X_0 := X - V(S, N)$. Finally Let $B$ be the union of all components of $G - (Y_1 \cup X_1 \cup X_2 \cup X_0)$ that intersect $V(G) - S_{b_1,b_2}$ and let $A := V(G) - (Y_1 \cup X_0 \cup X_1 \cup X_2 \cup B)$. Then $((A \cup B), (Y_1 \cup X_0 \cup X_1 \cup X_2))$ is a weakly balanced skew partition. □