A *convex corner* is a set $A \subseteq \mathbf{R}_+^n$ that is convex, compact, full dimensional and satisfying that $a \in A, a' \leq a$ implies $a' \in A$. For example, $\mathrm{STAB}(G)$ and $\mathrm{QSTAB}(G)$ are convex corners. The *antiblocker* of a convex corner A is $A^* := \{x \in \mathbf{R}_+^n : a \cdot x \leq 1 \text{ for } \forall a \in A\}$. For example,

$$\begin{aligned}
(\mathrm{STAB}(G))^* &= \{x \in \mathbf{R}_+^n : a \cdot x \leq 1, \forall a \in \mathrm{STAB}(G)\} = \\
&= \{x \in \mathbf{R}_+^n : a \cdot x \leq 1 \text{ for any incidence vector } a \text{ of a stable set}\} = \\
&= \{x \in \mathbf{R}_+^n : \sum_{v \in S} x_v \leq 1, \forall \text{stable set } S\} = \\
&= \mathrm{QSTAB}(\overline{G})
\end{aligned}$$

Clearly, $A \subseteq (A^*)^*$.

Exercise 17.5 Show that

$$(A^*)^* = A.$$

Denote

$$H_A(p) = \min_{x \in A} \sum_{i=1}^n p_i \log \frac{1}{x_i}.$$

Lemma 17.6 *Let $A, B \subseteq \mathbf{R}_+^n$ be convex corners. Then, $H_A(p) \leq H_B(p)$ for any probability distribution p if and only if $B \subseteq A$.*

Proof. $\Rightarrow$) trivial.
$\Leftarrow$) Assume $H_A(p) \leq H_B(p)$. Let $\Lambda(x) = (\log \frac{1}{x_1}, ..., \log \frac{1}{x_n})$. Now,

$$H_A(p) = \min_{x \in \Lambda(A)} p \cdot x.$$

Let $A' := \Lambda(A)$ and $B' := \Lambda(B)$. Then,

$$\min_{x \in A'} p \cdot x \leq \min_{x \in B'} p \cdot x$$

for every probability distribution p. This implies $B' \subseteq A'$ (exercise), and hence $B \subseteq A$, as desired. $\square$

Lemma 17.7 *If A, B are convex corners and $B \subseteq A^*$, then $H(p) \leq H_A(p) + H_B(p)$.*

Proof. Pick $a \in A$, $b \in B$ such that

$$H_A(p) = \sum_{i=1}^{n} p_i \log \frac{1}{a_i}, \text{ and } H_B(p) = \sum_{i=1}^{n} p_i \log \frac{1}{b_i}.$$

Since $b \in B \subseteq A^*$, $a \cdot b \leq 1$. Then,

$$H_A(p) + H_B(p) = \sum_{i=1}^{n} p_i \log \frac{1}{a_i b_i} \geq H(p).$$

$\square$

Theorem 17.8 *(Csiszár, Körner, Lovász, Marton, Simonyi and Lozanovski) TFAE for convex corners $A, B \subseteq \mathbf{R}_+^n$:*

i) $A^* \subseteq B$.

ii) *(A, B) is a generating pair: that is, any probability distribution p can be written as $p_i = a_i b_i$ for $a \in A, b \in B$.*

iii) *$H(p) \geq H_A(p) + H_B(p)$ for every probability distribution p.*

Corollary 17.9 $H(p) = H_A(p) + H_{A^*}(p)$.

Proof of Theorem 17.8. i) $\Rightarrow$ ii) Let $A^* \subseteq B$. Note

$$H_A(p) = \min_{x \in A} \sum_{i=1}^{n} p_i \log \frac{1}{x_i}.$$

Let

$$f(x) := \sum_{i=1}^{n} p_i \log \frac{1}{x_i}$$

and let a minimize f over all $x \in A$. Define b by

$$\begin{aligned} b_i &= p_i / a_i \text{ if } p_i > 0 \\ &= 0. \end{aligned}$$

Then b is well-defined, for $a_i = 0 \Rightarrow p_i = 0$. The sets A and $X := \{x \in \mathbf{R}_+^n : f(x) < f(a)\}$ are disjoint and so can be separated by a supporting hyperplane to X at a. But f is smooth and

$$\mathrm{grad}(-f)(a) = (p_1/a_1, p_2/a_2, ..., p_n/a_n) = b.$$

So the separating hyperplane is $\{x \in \mathbf{R}^n : b \cdot x = 1\}$. That means $b \cdot x \leq 1, \forall x \in A \Rightarrow b \in A^* \subseteq B$. So $p_i = a_i b_i, a \in A, b \in B$.

ii)$\Rightarrow$ iii) Suppose $p_i = a_i b_i, a \in A, b \in B$.

$$\begin{aligned} H(p) &= \sum p_i \log \frac{1}{p_i} = \sum p_i \log \frac{1}{a_i b_i} = \\ &= \sum p_i \log \frac{1}{a_i} + \sum p_i \log \frac{1}{b_i} \geq H_A(p) + H_B(p). \end{aligned}$$

iii)$\Rightarrow$ i) Notice $H(p) = H_A(p) + H_{A^*}(p)$ by Lemma 17.7 and the already proven implication i)$\Rightarrow$iii). Since $H_A(p) + H_B(p) \leq H(p) = H_A(p) + H_{A^*}(p)$, $H_B(p) \leq H_{A^*}(p)$ for $\forall p$ and therefore $A^* \subseteq B$ by Lemma 17.6.

Lemma 17.10 *(McDiarmid) For any two convex corners $A, B \subseteq \mathbf{R}_+^n$,*

$$\max_p \{H(p) - H_A(p) - H_B(p)\} = \log \max\{a \cdot b : a \in A, b \in B\}.$$

Proof. $\leq$) Let p be given and let $a \in A, b \in B$.

$$H(p) - H_A(p) - H_B(p) = \sum p_i \log \frac{a_i b_i}{p_i} \leq \log \sum_{i=1}^{n} p_i \frac{a_i b_i}{p_i} \leq \log(a \cdot b)$$

$\geq$) Given $a \in A, b \in B$, let

$$p_i = \frac{a_i b_i}{a \cdot b}$$

Then, $H(p) - H_A(p) - H_B(p) = \sum p_i \log \frac{a_i b_i}{p_i} = \sum p_i \log(a \cdot b) = \log(a \cdot b)$. □

Corollary 17.11

$$\max_p \{H_A(p) + H_B(p) - H(p)\} = \log \max\{a \cdot b : a \in A^*, b \in B^*\}.$$

Proof. We have $H(p) = H_A(p) + H_{A^*}(p)$.

$$\begin{aligned}\max_p\{H_A(p) + H_B(p) - H(p)\} &= \max_p\{H(p) - H_{A^*}(p) + H(p) - H_{B^*}(p) - H(p)\}\\ &= \log\max\{a \cdot b : a \in A^*, b \in B^*\}\end{aligned}$$

by Lemma 17.10. □

Corollary 17.12 *(Simonyi)*

$$\max_p\{H(G,p) + H(\overline{G},p) - H(p)\} = \log \mathrm{imp}(G).$$

Proof.

$$\begin{aligned} &\max_p\{H(G,p) + H(\overline{G},p) - H(p)\} = \\ =\ &\max\{H_{\mathrm{STAB}(G)}(p) + H_{\mathrm{STAB}(\overline{G})}(p) - H(p)\} = \\ =\ &\log\max\{a \cdot b : a \in \mathrm{STAB}^*(G), b \in \mathrm{STAB}^*(\overline{G})\} = \\ =\ &\log\max\{a \cdot b : a \in \mathrm{QSTAB}(\overline{G}), b \in \mathrm{QSTAB}(G)\} = \\ =\ &\log \mathrm{imp}(G)\end{aligned}$$

□

Corollary 17.13 *(Csiszár, Körner, Lovász, Marton and Simonyi) $H(G,p) + H(\overline{G},p) = H(p)$ for any probability distribution p if and only if G is perfect.*

G is *weakly spliting* if $H(G,p) + H(\overline{G},p) = H(p)$ for some probability distribution p that satisfies $p_v > 0$ for every $v \in V(G)$. G is *normal* if it has a set $\Im$ of stable sets and a set $\mathcal{Q}$ of cliques such that both $\Im$ and $\mathcal{Q}$ cover all vertices of G and $S \cap Q \neq \emptyset$ for every pair of $S \in \Im$ and $Q \in \mathcal{Q}$. For example, $C_5, C_7, \overline{C}_7$ are not normal.

Theorem 17.14 *(Körner and Longo) G is weakly splitting if and only if it is normal.*

Proof. $\Leftarrow$) Let $\Im$ and $\mathcal{Q}$ be as definition of normal. Pick nowhere zero probability distributions s and q on $\Im$ and $\mathcal{Q}$, respectively. Define

$$p_v = s_v \cdot q_v = \left(\sum_{S: v \in S \in \Im} s(S)\right) \cdot \left(\sum_{Q: v \in Q \in \mathcal{Q}} q(Q)\right)$$

Then,

$$\begin{aligned}\sum_{v\in V(G)} p_v &= \sum_{v\in V(G)} \left(\sum_{S:v\in S\in\mathfrak{S}} s(S)\right)\cdot\left(\sum_{Q:v\in Q\in\mathcal{Q}} q(Q)\right) = \\ &= \sum_{v\in V(G)}\sum_{v\in S\cap Q} s(S)q(Q) = \sum_{S,Q} s(S)q(Q)|S\cap Q| = \\ &= \sum_{S,Q} s(S)q(Q) = \sum_S s(S)\sum_Q q(Q) = 1\cdot 1 = 1.\end{aligned}$$

Since $p_v = s_v q_v$, by the proof of (ii)$\Rightarrow$(iii) in Theorem 17.8 we have

$$H(p) \geq H_{\mathrm{STAB}(G)}(p) + H_{\mathrm{STAB}(\overline{G})}(p) = H(G,p) + H(\overline{G},p) \geq H(p).$$

$\Rightarrow$) Postponed to next lecture.