**Lemma 17.15** *Let $A, B \subseteq \mathbf{R}^n_+$ be convex corners, let $p$ be a probability distribution on $[n]$ and let $a \in A, b \in B$ achieve the minima in $H_A(p), H_B(p)$, respectively. Assume $a \cdot b \leq 1$. Then $p_i = a_i b_i, \forall i = 1, ..., n \Leftrightarrow H(p) \geq H_A(p) + H_B(p)$.*

**Proof.** $\Rightarrow$)

$$H(p) = \sum_i p_i \log \frac{1}{p_i} = \sum_i p_i \log \frac{1}{a_i} + \sum_i p_i \log \frac{1}{b_i} = H_A(p) + H_B(p)$$

$\Leftarrow$)

$$\sum_i p_i \log \frac{1}{p_i} = H(p) \geq H_A(p) + H_B(p) = \sum_i p_i \log \frac{1}{a_i} + \sum_i p_i \log \frac{1}{b_i} = \sum_i p_i \log \frac{1}{a_i b_i}$$

The minimum of $\sum_i p_i \log \frac{1}{x_i}$ attained at $x_i = p_i$ and so $p_i = a_i b_i$. □

**Theorem 17.16** *(Körner,Longo) $H(G, p) + H(\overline{G}, p) = H(p)$ for some nowhere zero $p$ iff $G$ is normal(i.e., $\exists\, \Im = \{\text{stable sets } S\}$, $\mathcal{Q} = \{\text{cliques } Q\}$ such that both $\Im, \mathcal{Q}$ cover all vertices and $|S \cap Q| = 1$ for $\forall S \in \Im, Q \in \mathcal{Q}$.)*

**Proof.** $\Leftarrow$) Let $s, q$ be probability distributions on $\Im, \mathcal{Q}$, respectively. Set $p$ as

$$p_v = s_v q_v = \left( \sum_{v \in S \in \Im} s(S) \right) \left( \sum_{v \in Q \in \mathcal{Q}} q(Q) \right).$$

Note that $s \in \mathrm{STAB}(G), q \in \mathrm{STAB}(\overline{G})$ and $s \cdot q = 1$.

$$H(p) \geq H_{\mathrm{STAB}(G)}(p) + H_{\mathrm{STAB}(\overline{G})}(p) = H(G, p) + H(\overline{G}, p) \geq H(p).$$

$\Rightarrow$) Let $H(G, p) + H(\overline{G}, p) = H(p)$ for some nowhere zero $p$. Let $s \in \mathrm{STAB}(G)$ minimize $\sum p_i \log \frac{1}{s_i}$ and $q \in \mathrm{STAB}(\overline{G})$ minimize $\sum p_i \log \frac{1}{q_i}$. Let

$$s = \sum_{S \in \Im} s_S \mathbf{1}_S, \;\; q = \sum_{Q \in \mathcal{Q}} q_Q \mathbf{1}_Q$$

and let

$$s_v = \sum_{v \in S \in \Im} s_S, \;\; q_v = \sum_{v \in Q \in \mathcal{Q}} q_Q,$$

where $s_v > 0, q_v > 0$ for all $v$. By Lemma 17.15, $p_v = s_v q_v$. Note $s \cdot q \leq 1$.

$$\begin{aligned} 1 &= \sum_v p_v = \sum_v s_v q_v = \sum_{S,Q} |S \cap Q| s_S q_Q \leq \\ &\leq \sum_{S,Q} s_S q_Q = 1 \end{aligned}$$

Therefore, $|S \cap Q| \neq 0$. □

**A few comments on application to sorting (Kahn and Kim)**

Given a poset $P$, we want to find a given linear extension by asking questions of the form "Is $x < y$?" for incomparable elements $x, y$. How many questions do we need in the worst case? At least $\log(e(P))$, where $e(P)$ is the number of linear extensions of $P$.

**Theorem 17.17** *In any poset $P$ that is not a chain, there exist $x, y \in P$ incomparable such that*

$$\frac{3}{11} \leq P[x < y] \leq \frac{8}{11}.$$

**Conjecture 17.18** *same for*

$$\frac{1}{3} \leq P[x < y] \leq \frac{2}{3}.$$

This would be best possible, as the poset with three elements and one relation shows.

For a poset $P$, let $G_P$ be its comparability graph. Define

$$H(P) = H(G_P, \frac{1}{n}\mathbf{1}).$$

**Theorem 17.19** *(Kahn and Kim) For any poset $P$ on $n$ elements,*

$$n(\log n - H(P)) \geq \log e(P) \geq \max\{\log(n!) - nH(P), Cn(\log n - H(P))\}$$

**Theorem 17.20** *(Kahn and Kim) In any poset $P$ that is not a chain, there exist incomparable elements $x, y \in P$ such that*

$$H(P(x < y)) \geq H(P) + \frac{C}{n}.$$

$H(P)$ can be approximated in polynomial time. Note $H(P) \leq \log n$. In $\frac{n}{C}(\log n - H(P))$ ($= O(\log(e(P)))$ by Theorem 17.19) steps we will get a chain. In the proof they use entropy slitting for the comparability graph of the poset $P$: $H(P(x > y)) = H(\text{uniform}) - H(\overline{P(x > y)}, \text{uniform})$, where $\overline{P(x > y)}$ is complementary graph.