18 Computing chromatic number for perfect graphs

The original algorithm is due to Grötschel, Lovász and Schrijver. We will present an easier algorithm of Karger, Motwani and Sudan.

Lemma 18.1 *For every $n \geq k$ there exist k unit vectors $x_1, x_2, \ldots, x_k \in \mathbf{R}^n$ such that $x_i \cdot x_j = -1/(k-1)$ for every $i \neq j \in \{1, \ldots, k\}$.*

Proof. Define $x_1, x_2, \ldots, x_k$ as follows. Let the jth coordinate of x_i be equal to $-\sqrt{(k-1)/k}$ if $j = i$, $1/\sqrt{k(k-1)}$ if $j \leq k$ and $j \neq i$, and 0 otherwise. It is routine to verify that the required conditions are satisfied. $\square$

Lemma 18.2 *For every $n \geq k$ and every set of k unit vectors S in $\mathbf{R}^n$ there exist distinct $x, y \in S$ such that $x \cdot y \geq -1/(k-1)$.*

Proof. Let $S = \{x_1, x_2, \ldots, x_k\}$. We have $0 \leq (x_1 + x_2 + \ldots + x_k)^2 = \sum_{i=1}^{k} \|x_i\|^2 + \sum_{i \neq j} x_i \cdot x_j = k + \sum_{i \neq j} x_i \cdot x_j$. Therefore $\sum_{i \neq j} x_i \cdot x_j \geq -k$. This sum contains $k(k-1)$ elements. One of them has to have value greater then or equal to $-k/(k(k-1)) = -1/(k-1)$ $\square$

For a graph G with $n = |V(G)|$ define

$\mu(G) = inf\{t \mid \exists f : V(G) \to \mathbf{S}^{n-1} \subseteq \mathbf{R}^n\text{:}\ f(u)f(v) \leq t \ \ \forall uv \in E(G)\}.$

By (18.1) $\mu(G) \leq -1/(\chi(G)-1)$ as we can color G using k colors and map i-th color class to x_i. By (18.2) $\mu(G) \geq -1/(\omega(G)-1)$ as we can consider the images of the vertices of a maximum clique.

Therefore for perfect graphs $\mu(G) = -1/(\chi(G)-1)$ and it suffices to compute μ in polynomial time. Using semidefinite programming it is possible to approximate $\mu(G)$ within $1/(2n^2)$. If G is perfect, then $\mu(G) \in \{1, 1/2, \ldots, 1/(n-1)\}$ and therefore it is possible to determine exact value of $\mu(G)$ from the approximation.

To find a maximum clique we repeatedly delete vertices from G such that at each step $\omega(G \setminus v) = \omega(G)$, where v is the vertex deleted. If we can not delete any vertices, G is a clique.

It is not known how to find optimal coloring using this method of evaluation of $\chi(G)$.

19 Odd holes

The following four conjectures are due to Gyárfás.

Conjecture 19.1 *There exists a function f such that $\chi(G) \le f(\omega(G))$ for every graph G.*

If in (19.1) we also exclude antiholes then by the SPGC the identity function satisfies the requirements. It is easy to see that $f(2) = 2$ as every triangle-free graph without odd holes is bipartite.

Theorem 19.2 (Robertson, Seymour, Thomas) $f(3) = 4$.

The tight example for (19.2) is provided by $\bar{C}_7$. It is not known if $f(4)$ exists.

Theorem 19.3 (Hoang, McDiarmid, Randerath and Schiermeyer) *For every $c \in \mathbf{Z}_+$ there exists graph G with no odd holes such that $\chi(G) > c\omega(G)$.*

Conjecture 19.4 *For every integer k there exists a function f_k such that $\chi(G) \le f_k(\omega(G))$ for every graph G with no hole of length greater then or equal to k.*

Conjecture 19.5 *For every integer k there exists a function f_k such that $\chi(G) \le f_k(\omega(G))$ for every graph G with no odd hole of length greater then or equal to k.*

Conjecture 19.6 *For every forest F there exists a function f_F such that $\chi(G) \le f_F(\omega(G))$ for every graph G with no induced subgraph isomorphic to F.*

Graph G is called 2-*divisible* if for every induced subgraph H of G there exists a partition (X, Y) of $V(H)$ such that $\omega(X), \omega(Y) < \omega(H)$. Every perfect graph is 2-divisible as there exists maximum independent set in it meeting every maximum clique. For every $k \ge 2$ the graph C_{2k+1} is not 2-divisible. The graph $\bar{C}_7$ is 2-divisible, because $\chi(\bar{C}_7) = 4$, $\omega(\bar{C}_7) = 3$ and we can partition $V(\bar{C}_7)$ into two sets containing two color classes each.

Conjecture 19.7 (Hoang, McDiarmid) *A graph G is 2-divisible if and only if it has no odd hole.*

If $\omega(G) \leq 3$ conjecture (19.7) follows from results for (19.1). It also implies (19.1) in general.

Open problem: Is there a polynomial time algorithm to test if a graph has an odd hole?

Note that there exists a polynomial time algorithm to test if a graph is Berge.

Theorem 19.8 (Conforti, Cornuéjols, Kapoor, Vuskovic) *There exists a polynomial time algorithm to test if a graph has an even hole*

20 Graphs with no holes

A graph is called *chordal* if it has no holes.

Theorem 20.1 *In a chordal graph every minimal cutset is a clique.*

Proof. By contradiction. Let C be a minimal cutset in G, let A_1 and A_2 be two components of $G \setminus C$ and let $x, y \in C$, $xy \notin E(G)$. Let P_i for $i = 1, 2$ be an induced path with ends x and y and all internal vertices in A_i. It is easy to verify that $P_1 \cup P_2$ induces a hole in G. $\square$

A vertex in a graph is called *simplicial* if its neighborhood is a clique.

Theorem 20.2 *Every chordal graph G has a simplicial vertex.*

Proof. By induction. The statement of the theorem is true if G is a clique. If a graph is not a clique we prove by induction that it contains two non adjacent simplicial vertices. If S is a cutset of G, and A_1 and A_2 are two components of $G \setminus S$ then by induction hypothesis $G[S \cup A_i]$ contains a simplicial vertex $x_i \in V(A_i)$. It follows that G contains non adjacent simplicial vertices x_1 and x_2. $\square$

Theorem (20.2) allows to test if a graph is chordal in polynomial time, since if $v \in V(G)$ is simplicial then G is chordal if and only if $G \setminus v$ is chordal.

A graph is called *weakly chordal* if it has no holes or antiholes of length at least 5. Note that every chordal graph is weakly chordal as every antihole of length at least 6 contains a hole of length 4.

Lemma 20.3 *Let G be weakly chordal, let $S \subseteq V(G)$ be a minimal cutset such that $G[S]$ is anticonnected and let A be a component of $G \setminus S$. Then A has an S-complete vertex.*

Proof. By induction on $|S|$. If $|S| = 1$ then lemma clearly holds. Pick $s_1, s_2 \in S$ adjacent, such that $G[S \setminus s_1]$ and $G[S \setminus s_2]$ are anticonnected. Denote an induced antipath in $G[S]$ between s_1 and s_2 by P. By induction hypothesis there exists $a_i \in V(A)$ complete to $S \setminus s_i$ for $i = 1, 2$. We may assume that $a_i s_i \notin E(G)$. We may also assume that $a_1 a_2 \in E(G)$ as otherwise graph induced by a_1, a_2 and $V(P)$ is an antihole. Let B be a component of $G \setminus S \setminus A$. If some $b \in V(B)$ is complete to S then b, a_1, a_2 and $V(P)$ induce an antihole. Therefore by induction hypothesis we can find $b \in V(B)$ complete to $S \setminus s_1$ and $b_1 s_1 \notin E(G)$. Then a_2, b_1 and $V(P)$ induce an antihole in G. $\square$

An *even pair* G is a pair of vertices x, y such that every induced x, y is even. Let x, y be an even pair in G and let G' be obtained from G by identifying x and y. Then G is Berge if and only if G' is Berge, G is perfect if and only if G' is perfect, $\omega(G) = \omega(G')$, $\chi(G) = \chi(G')$. Note that not every Berge graph has an even pair. The line graph of $K_{3,3}$ does not. An even pair x, y is called a *2-pair* if every induced x, y path has length two.

Theorem 20.4 (Hayward, Hoang, Maffray) *Every chordal graph G is a clique or has a 2-pair.*

Proof. We claim that every minimum cutset of a weakly chordal graph that is not a clique includes a 2-pair. We prove this statement and statement of the theorem simultaneously by induction on $|V(G)|$. If G is a clique both statements hold. Otherwise let S be a minimal cutset of G.

First, suppose S is a clique. Let A be a component of $G \setminus S$. By induction hypothesis A has a 2-pair x, y, unless A is a clique. Therefore we may assume that every component of $G \setminus S$ is a clique. Every two vertices from different components of $G \setminus S$ form a 2-pair in G.

Now suppose S is not a clique and is not anticonnected. There exists a partition of S into D_1 and D_2, such that D_1 is not clique and D_1 is complete to D_2. By induction there exists a 2-pair $x, y \in V(D_1)$ in $G \setminus D_2$. It is easy to check that x and y form a 2-pair in G.

It remains to consider the case when S is anticonnected. By (20.3) each component of $G \setminus S$ contains a vertex complete to S. If every component of $G \setminus S$ is a singleton then a 2-pair which exists in $G[S]$ by induction hypothesis will be a 2-pair in G. Therefore we may assume that some component A of $G \setminus S$ contains at least 2 vertices. Consider $A \setminus a$ and let B be component of it. Assume $N(B)$ is not a clique. Let $S' = N(B) \setminus A$. We may assume that B is chosen so that S' is a minimal cutset in $G \setminus a$. By induction hypothesis there exists a 2-pair $x, y \in V(S')$ in $G \setminus a$, which is also a 2-pair in G.

Therefore we may assume $N(B)$ is a clique. By induction hypothesis there exists a 2-pair $x, y \in V(S)$ in $G \setminus B$, which again is a 2-pair in G as no induced path of length two connecting x to y can include a vertex of B. $\square$

Note that if in a graph G we put an edge between the vertices of a 2-pair to produce a graph G', then G is weakly chordal if and only if G' is. We can repeat this process to produce a clique from any weakly chordal graph. This is an efficient way to test if a graph is weakly chordal.