Graph Planarity and Related Topics

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Abstract. This compendium is the result of reformatting and minor editing of the author's transparencies for his talk delivered at the conference. The talk covered Euler's Formula, Kuratowski's Theorem, linear planarity tests, Schnyder's Theorem and drawing on the grid, the two paths problem, Pfaffian orientations, linkless embeddings, and the Four Color Theorem.

1 BASICS

A *plane* graph is a graph drawn in the plane with no crossings. A graph is *planar* if it can be drawn in the plane with no crossings. Edges can be represented by homeomorphic images of $[0,1]$, continuous images of $[0,1]$, piecewise-linear images of $[0,1]$, or, by Fáry's theorem [11], by straight-line segments, and the class of planar graphs remains the same.

EULER'S FORMULA. For a connected plane graph G, $|V(G)| + |F(G)| = |E(G)| + 2$.

COROLLARY. If a planar graph G has at least three vertices, then $|E(G)| \leq 3|V(G)| - 6$. If, in addition, G has no triangles, then $|E(G)| \leq 2|V(G)| - 4$.

KURATOWSKI'S THEOREM. A graph is planar if and only if it has no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$.

The "only if" part follows easily from the fact that K_5 and $K_{3,3}$ have too many edges. The "if" part is much harder. An elegant proof was found by Thomassen [40].

2 TESTING PLANARITY IN LINEAR TIME

There are two classical linear-time planarity tests by Hopcroft and Tarjan [13], and by Lempel, Even and Cederbaum [19] and Booth and Lueker [7]. I will outline a new algorithm, found independently by Shih and Hsu [36] and Boyer and Myrvold [8]. A self-contained description of the algorithm may be found in [37].

Outline. Find a DFS tree T, and number its vertices $v_1, v_2, \ldots, v_n$ so that the numbers are decreasing along every path of T from the root, and the graph induced by the vertices $v_i, v_{i+1}, \ldots, v_n$ is connected for all $i = 1, 2, \ldots, n$. The algorithm recursively adds the vertices $v_1, v_2, \ldots, v_n$ in the order listed. At the beginning of the ith iteration we have a certain "tree-structure" illustrated in Figure 1.

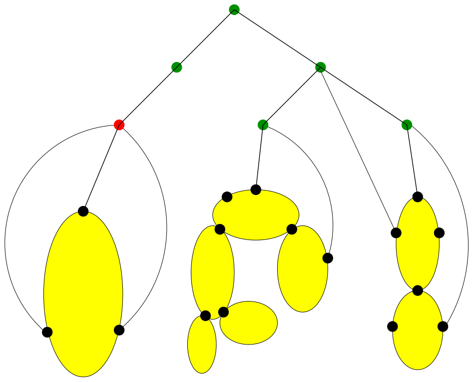

Figure 1.

The shaded ovals represent blocks of the graph induced by $v_1, v_2, \ldots, v_{i-1}$. Each of these blocks B is embedded in the plane, and all edges joining it to the rest of the graph have one end incident with the infinite region of B. We now add the vertex v_i to this structure; it can be shown that if this cannot be done, then the graph has a K_5 or $K_{3,3}$ subdivision.

3 SCHNYDER'S THEOREM and DRAWING ON A GRID

THEOREM. (Schnyder [31]) A graph is planar if and only if its vertex-edge poset has dimension at most three.

The latter is equivalent to the existence of linear orderings $\leq_1, \leq_2, \leq_3$ satisfying:
(∗) $\forall uv \in E(G)\ \forall w \in V(G) - \{u, v\}\ \exists i$ such that $u \leq_i w$ and $v \leq_i w$.

Given $\leq_1, \leq_2, \leq_3$ there exists a 1-1 map $v \in V(G) \to (v_1, v_2, v_3) \in \mathbf{R}^3$ such that
(i) $v_1 + v_2 + v_3 = 2n - 5$ for all $v \in V(G)$
(ii) $v_i \in [0, 2n - 5]$ is an integer
(iii) for $uv \in E(G)$ we have $u \leq_i v$ if and only if $u_i \leq v_i$

and the coordinatewise orderings satisfy (∗). It follows that this gives a straight-line embedding on a grid. The latter was independently obtained by de Fraysseix, Pach and Pollack [12], and since then has been the subject of extensive research.

4 COLIN de VERDIERE'S PARAMETER

Let $\mu(G)$ be the maximum corank of a matrix M satisfying
(i) for $i \neq j$, $M_{ij} = 0$ if $ij \notin E$ and $M_{ij} < 0$ otherwise,
(ii) M has exactly one negative eigenvalue,
(iii) if X is a symmetric $n \times n$ matrix such that $MX = 0$ and $X_{ij} = 0$ whenever $i = j$ or $ij \in E$, then $X = 0$.

THEOREM. (Colin de Verdière [10]) $\mu(G) \leq 3$ if and only if G is planar.

5 SEPARATORS

Let G be a graph on n vertices. A *separator* in G is a set S such that every component of $G \backslash S$ has at most $2n/3$ vertices.

THEOREM. (Lipton, Tarjan [20]) Every planar graph has a separator of size at most $\sqrt{8n}$.

Alon, Seymour and Thomas [2] found a simpler proof and improved the constant to $\sqrt{4.5n}$. In [1] they proved that graphs not contractible to K_t have a separator of size at most $\sqrt{t^3 n}$.

6 THE TWO PATHS PROBLEM

Given a graph G and $s_1, s_2, t_1, t_2 \in V(G)$ do there exist disjoint paths P_1, P_2 in G such that P_i has ends s_i and t_i?

A reduction. Let $V(G) = A \cup B$, where $|A \cap B| \leq 3$, no edge has one end in $A - B$ and the other in $B - A$, $s_1, s_2, t_1, t_2 \in A$, and, subject to that, $|A \cap B|$ is minimum. Let H be obtained from G by deleting $B - A$ and adding an edge joining every pair of vertices in $A \cap B$. Then the paths exist in G if and only if they exist in H. We say that G is *reduced* if $|V(G)|$ cannot be lowered by performing this operation.

THEOREM. (Seymour [33], Shiloach [35], Thomassen [39]) If a graph G is reduced, then the paths exist if and only if G cannot be drawn in a disk with s_1, s_2, t_1, t_2 drawn on the boundary of the disk in order.

7 PFAFFIAN ORIENTATIONS

An orientation of a graph G is *Pfaffian* if every even cycle C such that $G\backslash V(C)$ has a perfect matching has an odd number of edges directed in either direction of the cycle. If a graph has a Pfaffian orientation, then the number of perfect matchings can be computed in polynomial time.

THEOREM. (Kasteleyn [16],[17]) Every planar graph has a Pfaffian orientation.

The decision problem whether a bipartite graph has a Pfaffian orientation is equivalent to:

- Pólya's permanent problem [24]
- the even directed cycle problem [34], [41], [43], [44]
- a hypergraph problem [32]
- the sign nonsingular matrix problem [9], [18], [42].

THEOREM. (McCuaig [23], Robertson, Seymour and Thomas [29]) A bipartite graph has a Pfaffian orientation if and only if it can be obtained from planar bipartite graphs and the Heawood graph by means of 0-, 2- and 4-sums.

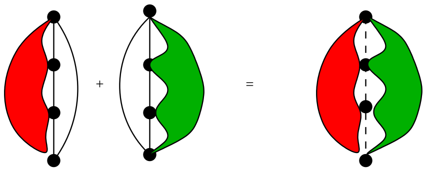

Figure 2. 4-sum.

COROLLARY. There exists a cubic algorithm to solve the above-mentioned problems.

8 LINKLESS EMBEDDINGS

A (piecewise linear) embedding of a graph in $\mathbf{R}^3$ is *linkless* if every two disjoint cycles have zero linking number.

THEOREM. (Robertson, Seymour and Thomas [28]) For a graph G the following conditions are equivalent:
(i) G has a linkless embedding

(ii) G has no subgraph that can be contracted onto one of seven specific graphs (K_6, $K_{3,3}$+vrtx, $K_{4,4}\backslash e$, ..., Petersen)
(iii) G has a flat embedding

An embedding of a graph G in $\mathbf{R}^3$ is *flat* if every cycle bounds a disk disjoint from the rest of G.

REMARK. If G is planar, then an embedding $G \hookrightarrow \mathbf{R}^3$ is flat if and only if the image of G is a subset of an embedded sphere.

THEOREM. (Robertson, Seymour and Thomas [28]) An embedding of a graph G in $\mathbf{R}^3$ is flat if and only if the fundamental group of the complement of every subgraph is free.

THEOREM. (Robertson, Seymour and Thomas [28]) If a 4-connected graph has a flat embedding, then it has a unique flat embedding. More generally, any two flat embeddings of the same graph are related by "3-switches".

THEOREM. (Lovász and Schrijver [21]) A graph G has a flat embedding if and only if $\mu(G) \leq 4$.

An embedding of a graph G in $\mathbf{R}^3$ is *knotless* if every cycle of G bounds a disk. No characterization of knotless graphs is known, but the following is a consequence of the Robertson-Seymour theory [26], [27].

THEOREM. There exists a cubic algorithm to test whether an input graph has a knotless embedding.

The above only guarantees the existence of an algorithm. In fact, at present no explicit algorithm (let alone a polynomial-time one) to decide whether a graph has a knotless embedding is known.

9 A DIGRESSION

The cross-product is not associative, and so

$$v_1 \times v_2 \times \cdots \times v_n \tag{1}$$

is not well-defined if $n > 2$. If we put in enough brackets to make it well-defined, we get a *bracketing*. For example, $((v_1 \times v_2) \times (v_3 \times v_4)) \times v_5$ is a bracketing.

THEOREM. (Kauffman [14]) For every two bracketings of (1) there exists an assignment of the standard unit vectors in $\mathbf{R}^3$ to $v_1, \ldots, v_n$ such that the evaluations of the two bracketings are equal and nonzero.

THEOREM. (Kauffman [14]) The above theorem is equivalent to the Four-Color Theorem.

10 THE FOUR-COLOR THEOREM

THEOREM. Every planar graph is 4-colorable.

COMMENTS:
• Simple statement, yet proof is long.
• A 1976 proof by Appel and Haken [3], [4], reprinted in [5].
• A simpler proof by Robertson, Sanders, Seymour and Thomas [25].
• Both proofs are computer assisted.
• There are over two dozen equivalent formulations (in terms of graphs [30], vector cross products [14], Lie algebras [6], divisibility [22], Temperley-Lieb algebras [15])
• There are interesting conjectured generalizations.
• See [38] for a survey.

11 OUTLINE OF a PROOF OF the 4CT

Let G be a counterexample with $|V(G)|$ minimum. It can be shown that G is an internally 6-connected triangulation, where a graph G is *internally* 6*-connected* if it is 5-connected, and for every vertex cut X of size five, $G\backslash X$ has exactly two components, one of which has only one vertex.

A *configuration* is a pair consisting of a near-triangulation K (i.e., a planar graph such that every region except possibly the infinite one is bounded by a cycle of length three) and a labeling of the vertices of K by integers. A configuration K *appears* in a triangulation T if K is an induced subgraph of T and for every vertex of K its label equals its degree in T. We exhibit a set $\mathcal{U}$ of 633 configurations such that the following two theorems hold. Some configurations in this set are depicted in Figure 3, where the labeling is indicated by vertex shapes—a solid circle stands for 5, a dot (or what appears as no symbol at all) for 6, a hollow circle for 7 and a hollow square for 8.

THEOREM 1. No member of $\mathcal{U}$ appears in a minimal counterexample to the 4CT.

THEOREM 2. For every internally 6-connected triangulation T, some member of $\mathcal{U}$ appears in T.

Thus Theorems 1 and 2, together with the fact that every minimal counterexample to the 4CT is an internally 6-connected triangulation, imply the Four-Color Theorem. The proof of Theorem 2 proceeds as follows. In a triangulation T with n vertices and e edges we have $e = 3n - 6$ and so

$$d_1 + d_2 + \cdots + d_n = 2e = 6n - 12,$$

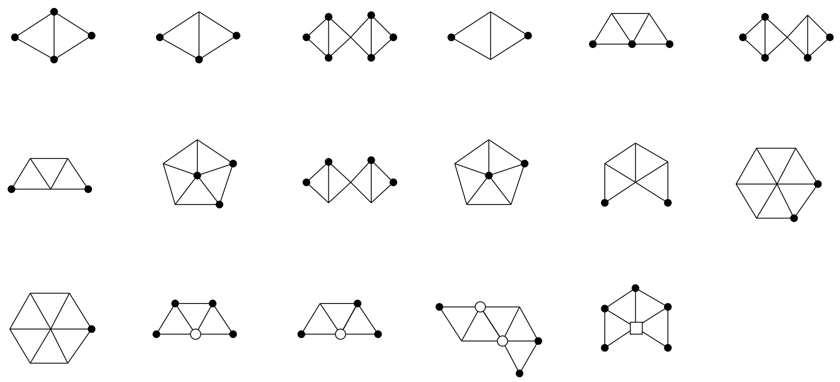

Figure 3. Examples of configurations.

where $d_1, d_2, \ldots, d_n$ are the degrees of vertices of T. This can be rewritten as

$$(6 - d_1) + (6 - d_2) + \cdots + (6 - d_n) = 12.$$

Initially, a vertex of degree d will receive a *charge* of $6 - d$. Thus the sum of the charges is 12. Charges will be redistributed according to certain rules, but the sum will remain the same. Thus there is a vertex v of positive charge. We show that a reducible configuration appears in the second neighborhood of v.

COROLLARY. There is a quadratic algorithm to 4-color planar graphs.

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