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Math 3012 – Applied Combinatorics Lecture 12

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Planar Graphs

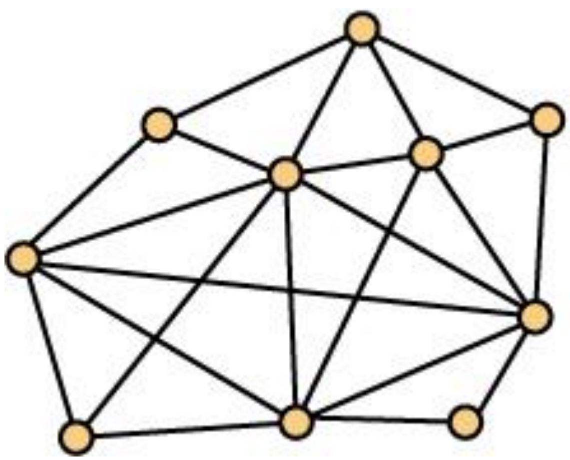
Definition A graph G is planar if it can be drawn in the plane with no edge crossings.

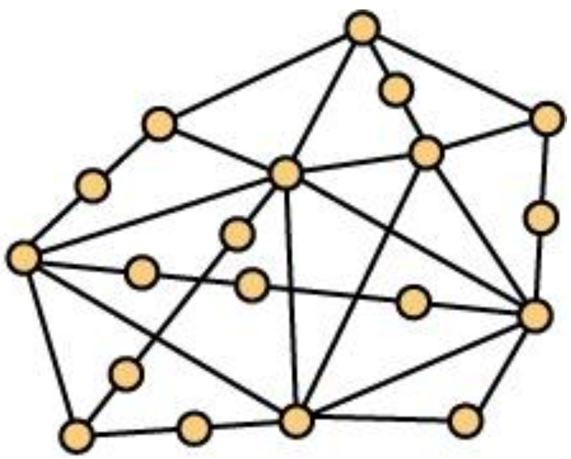
Theorem (Euler's Formula) If n, q and f denote respectively, the number of vertices, edges and faces in a plane drawing of a planar graph with t components, then

$$n - q + f = 1 + t$$

Homeomorphs of a Graph

Definition A graph H is a homeomorph of a graph G if H is obtained by "inserting" one or more vertices on some of the edges of G. The graph on the right is a homeomorph of the graph on the left.





The Maximum Number of Edges in a Planar Graph

Theorem If G is a planar graph with $n \geq 3$ vertices and q edges, then $q \leq 3n - 6$.

Theorem The complete graph K_5 is non-planar.

Observation If a graph G is planar, then any subgraph of G is planar.

Observation If a graph H is a homeomorph of a graph G, then H is planar if and only if G is planar.

Consequence A graph is non-planar if it contains a homeomorph of the complete graph K_5 as a subgraph.

Two-Colorable Planar Graphs

Theorem If G is a 2-colorable planar graph with $n \geq 3$ vertices and q edges, then $q \leq 2n - 4$.

Proof Fix the value of n and consider a plane drawing of a planar graph G on n vertices having the maximum number of edges. Clearly, G is connected and has no bridges so that every edge of G belongs to exactly two faces.

For each *even* $m \geq 4$, let f_m be the number of faces whose boundary is a cycle of size m.

Using Euler to Determine Non-Planarity (2)

Theorem The complete bipartite graph $K_{3,3}$ is non-planar.

Proof The complete bipartite graph $K_{3,3}$ is 2-colorable and has $n = 6$ vertices and $q = 9$ edges. Since $9 > 2\cdot6 - 4 = 12 - 4 = 8$, $K_{3,3}$ cannot be planar.

Kuratowski's Theorem

Theorem (Kuratowski, '30) *A graph is non-planar if and only if it contains a homeomorph of the complete graph K_5 or a homeomorph of the complete bipartite graph $K_{3,3}$ as a subgraph.*

Remark Highly efficient algorithms for planarity testing are known, and they have running time which is linear in the input size. This issue remains an active research topic. These algorithms use advanced data structures and are beyond the level of our course.

Maximum Clique Size and Chromatic Number

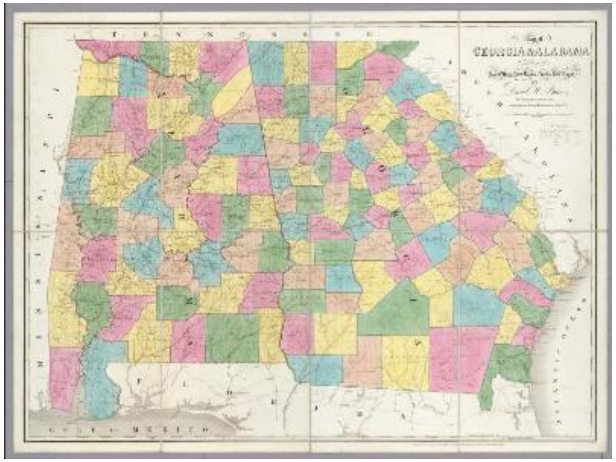
Fact Since the complete graph K_5 is non-planar, if G is a planar graph, then it has maximum clique size at most 4.

Note The following result, known as the "four color theorem" has a history spanning more than 100 years.

Theorem If G is a planar graph, then the chromatic number of G is at most 4, i.e., G can be 4-colored.

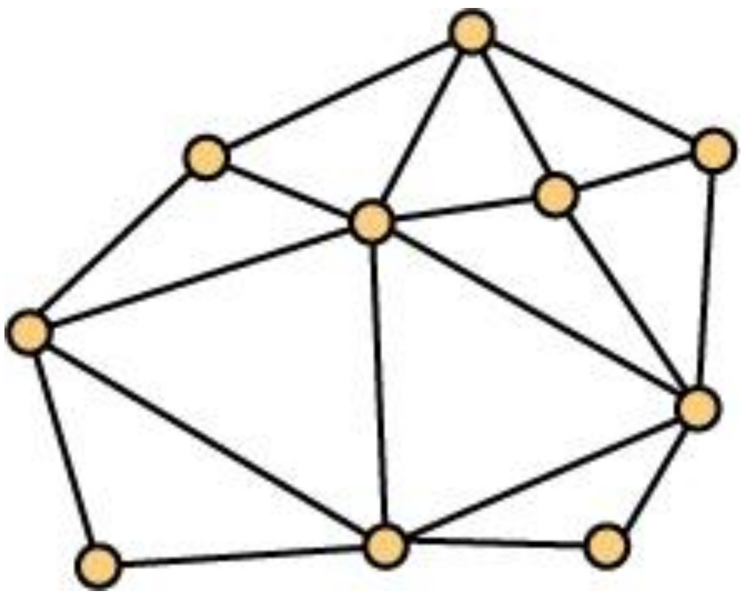
Coloring Planar Maps

Historical Note The problem of coloring a planar map so that states/countries/regions sharing a common boundary have different colors is a problem with a several hundred year history. The map shown below is the state of Georgia and is due to David H. Burr (1803 – 1875). Note that it has been 5-colored, so Mr. Burr could have done better!!!



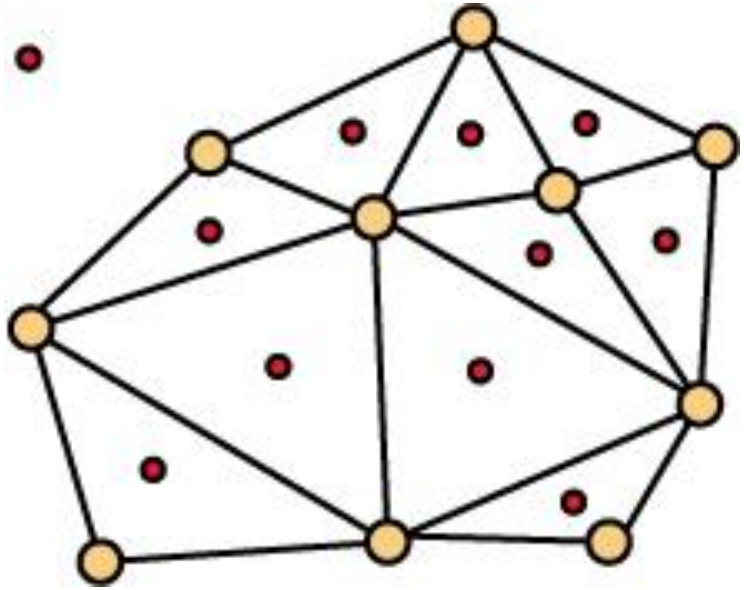
Planar Graphs and Planar Maps (1)

Observation A planar graph has a dual graph which is also planar. In the dual graph, the concepts of faces and vertices are interchanged.



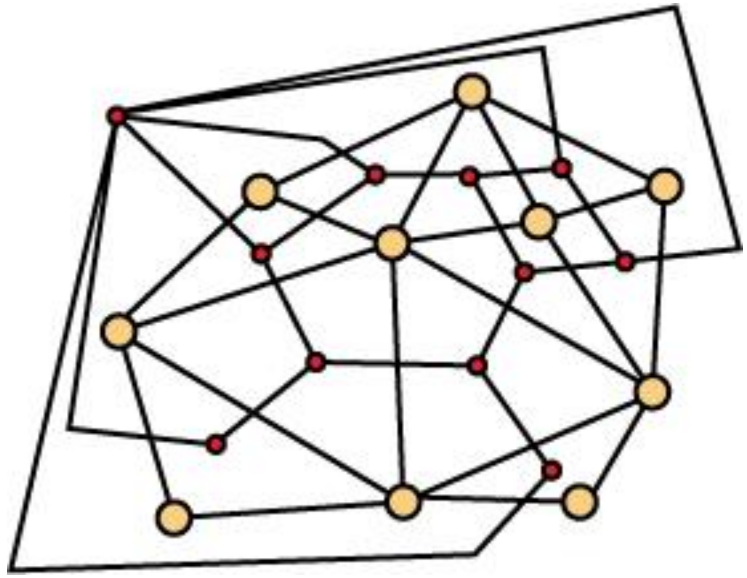
Planar Graphs and Planar Maps (2)

Observation Insert a “capital city” in each region. Don't forget the infinite region.



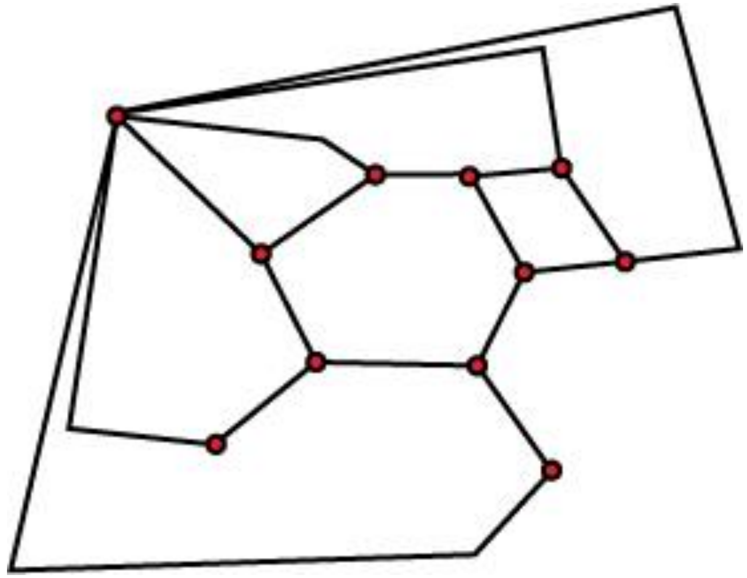
Planar Graphs and Planar Maps (3)

Observation Join two capitals with an edge when their respective regions share a boundary edge.



Planar Graphs and Planar Maps (4)

Observation Remove the original graph to obtain the dual graph.



The Four Color Theorem

Theorem (Appel and Haken, 1977) If G is a planar graph, then G can be 4-colored.

Historical Note The proof remains a controversial issue … for two reasons. First, it used extensive computing to verify certain claims. Second, some researchers are not convinced that the paper and pencil reductions to the computational stage are complete and correct.

The Four Color Theorem (2)

Follow-up Note Robertson, Sanders, Seymour and Thomas, 1996, have given a definitive, albeit still computer based proof, of the Four Color Theorem. The correctness of their programs has been verified independently by multiple sources. You can find a number of fascinating stories about this problem by doing a web search … but a word of caution that the final chapter in this story has yet to be written!! Nevertheless, the basic approach of Appel and Haken has been validated … and RSST were clear in their work about this fact.

Game Coloring for Graphs

Definition The game chromatic number of a graph is the least positive integer t for which there is a strategy for Alice that will enable her, working in "cooperation" with Bob, to color the graph using t colors and alternating turns.

Note The issue as to who goes first can be important.

Theorem (Kierstead and Trotter, '94) The game chromatic number of a planar graph is at most 33.

Two Challenging Exercises

Observation The chromatic number of a tree is two if it has an edge. However, the game chromatic number of a tree is at most 4 and this result is best possible. This is a good exercise for a senior level undergraduate course in graph theory.

Follow-Up Note Kierstead and Zhu have been carrying on a running competition for 20 years, and it is now known that the game chromatic number of a planar graph is at most 17 with Zhu in the winning position for now. From below, a lower bound of 7 is known. If you really want to get an A+++, move either bound.

List Colorings of Graphs

Definition The list chromatic number of a graph is the smallest integer t so that a proper coloring of the graph can always be found using colors from prescribed lists of size t, one list for each vertex. Note that different vertices can have different lists.

Example When $n = C(2t-1, t)$, the complete bipartite graph $K_{n,n}$ has list chromatic number $t + 1$.

Theorem (Thomasen, 1994) The list chromatic number of a planar graph is at most 5.