October 20, 2015

Math 3012 – Applied Combinatorics Lecture 17

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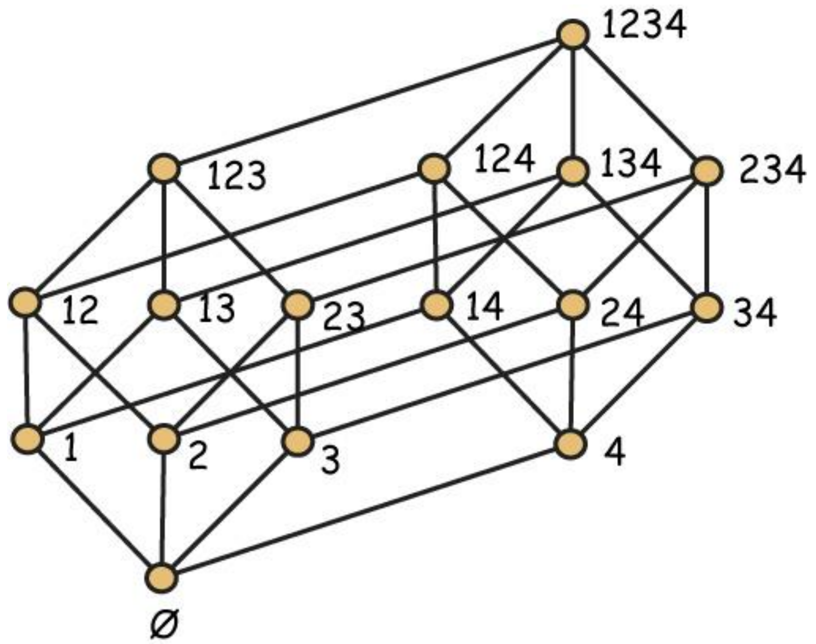
Reminder

Test 2 Thursday, October 22, 2015. Details on material for which you will be responsible were sent by email this last weekend. Study hard. Experience shows that the middle portion of this course has considerably more depth and significance than the first part.

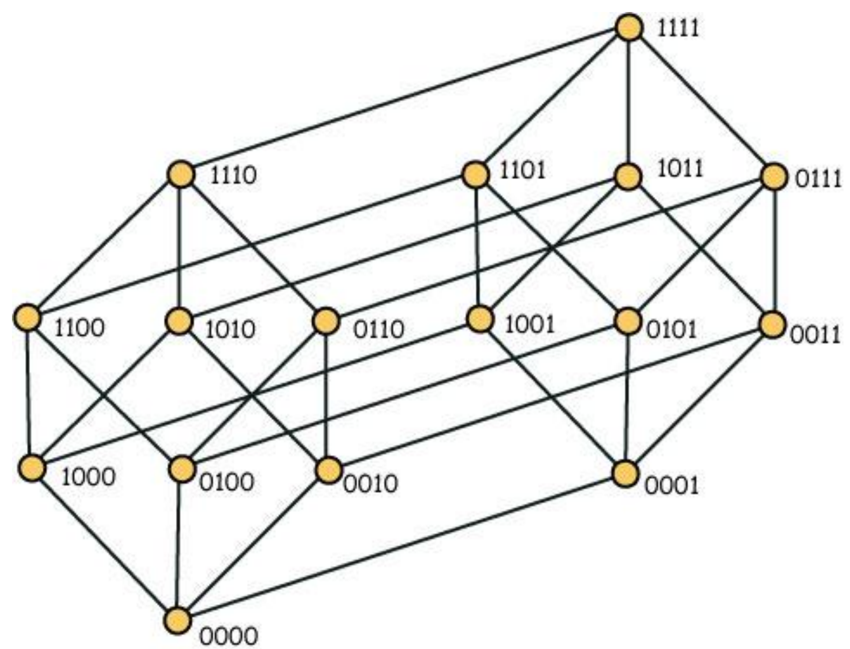
Subset Lattices

Definition For an integer $n \geq 1$, the poset consisting of all subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion is called the subset lattice. We will denote it as $\mathbf{2}^n$. Here is $\mathbf{2}^4$.



Subset Lattices - Cubes

Remark Using the alternate notation for subsets as bit strings, subset lattices are also called cubes. Here is the 4-cube.



Basic Properties of Subset Lattices

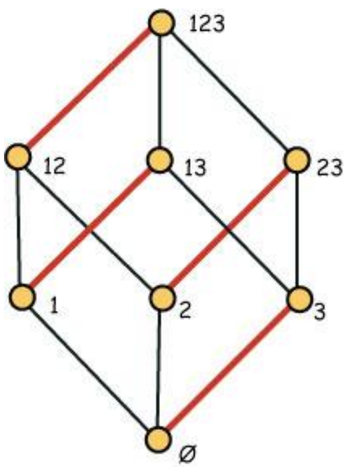
Fact The size of $\mathbf{2}^n$ is 2^n.

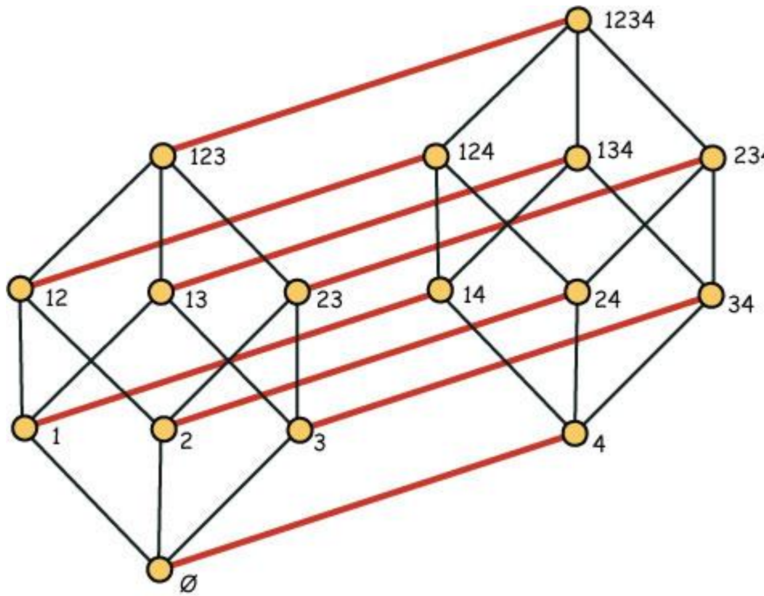
Fact The unique maximal element in $\mathbf{2}^n$ is the set $\{1, 2, \ldots, n\}$ and the unique minimal element is the empty set $\emptyset$.

Fact The height of $\mathbf{2}^n$ is $n + 1$. In fact, all maximal chains are maximum.

Inductive Nature of Subset Lattices

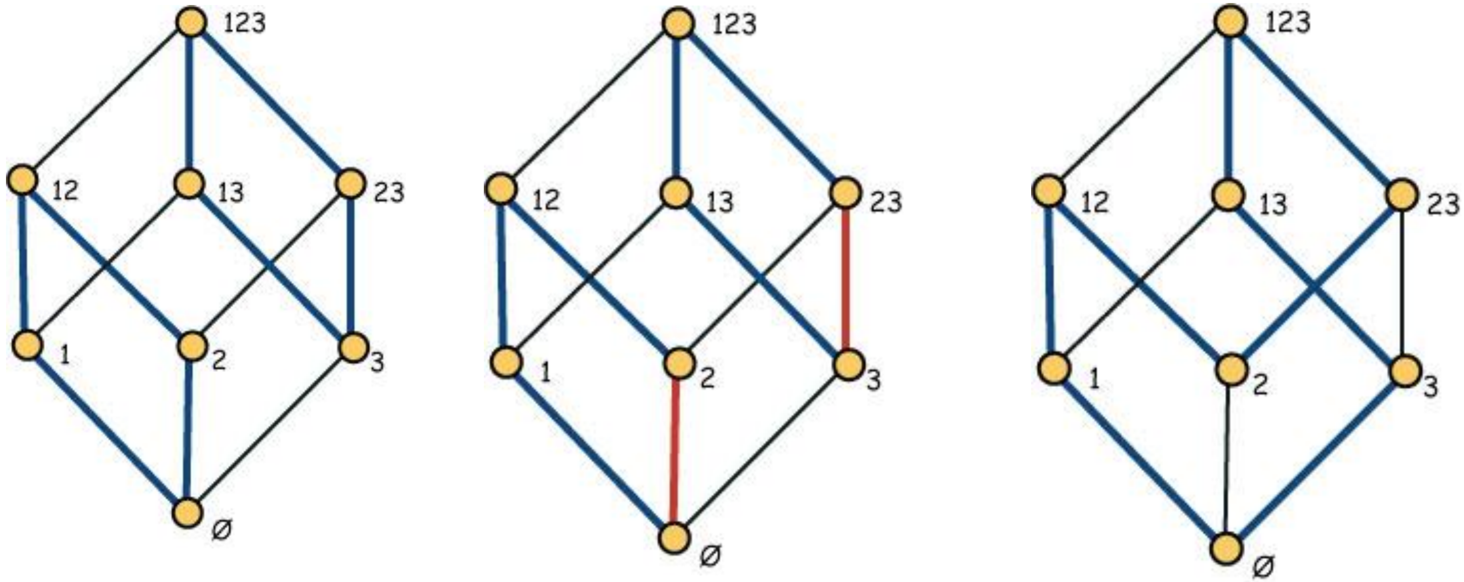
Basic Fact The subset lattice $\mathbf{2}^{n+1}$ can be viewed as $\mathbf{2} \times \mathbf{2}^n$.





Hamiltonian Property for Subset Lattices

Theorem For $n \geq 2$, the n-cube subset $\mathbf{2}^n$ is Hamiltonian.



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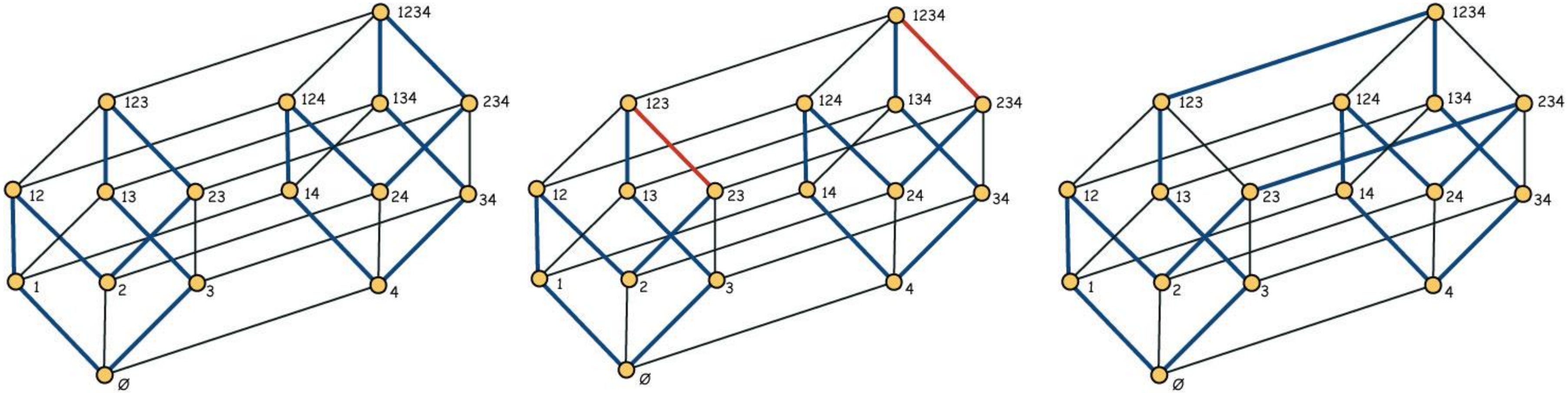


The Width of Subset Lattices

Fact If A is a set with $|A| = k$, then the number of maximal chains in $\mathbf{2}^n$ containing A is $k!\,(n-k)!$

Fact The width of $\mathbf{2}^n$ is at least as large as any binomial coefficient $C(n, k)$, where $0 \le k \le n$.

Fact The largest binomial coefficient of the form $C(n, k)$ is when $k = \lfloor n/2 \rfloor$. When n is even, there is just one value of k for which $C(n, k)$ is maximum. When n is odd, there are two. For example, the width of $\mathbf{2}^{13} \ge C(13, 6) = C(13, 7)$ while the width of $\mathbf{2}^{14} \ge C(14, 7)$.

The Width of Subset Lattices (2)

Theorem Fact (Sperner, '28) The width of the subset lattice $\mathbf{2}^n$ is the binomial coefficient $C(n, \lfloor n/2 \rfloor)$.

Note We will give two proofs of this result in class. The first proof is the more classical of the two and rests on the following elementary fact.

Fact If A is a subset of $\{1, 2, \ldots, n\}$ and $|A| = k$, then the number of maximal chains containing A is $k!\,(n - k)!$ To see this, consider bit strings. There are $k!$ ways to add the bits in A and then another $(n - k)!$ ways to add the bits in the complement of A.

The Width of Subset Lattices (3)

Proof of Spener's theorem Let $\{A_1, A_2, \ldots, A_t\}$ be a maximum antichain in $\mathbf{2}^n$. For each i, let $k_i = |A_i|$. Then

$$\sum_{1 \le i \le t} k_i! \, (n - k_i)! \le n!$$

$$\sum_{1 \le i \le t} [k_i! \, (n - k_i)! \,]/n! \le 1.$$

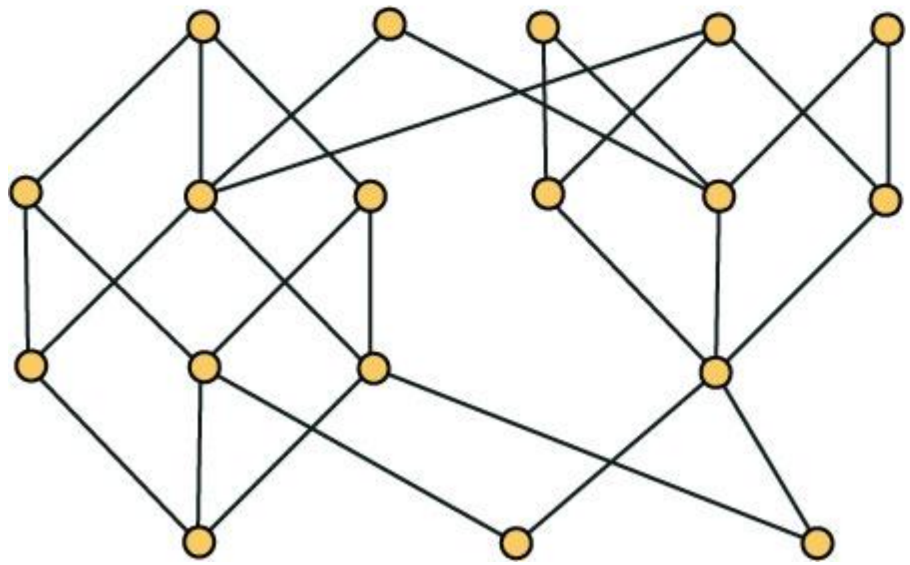
$$\sum_{1 \le i \le t} 1/C(n, k_i) \le 1$$

$$t \,/ C(n, \lfloor n/2 \rfloor) \le 1$$

$$t \le C(n, \lfloor n/2 \rfloor)$$

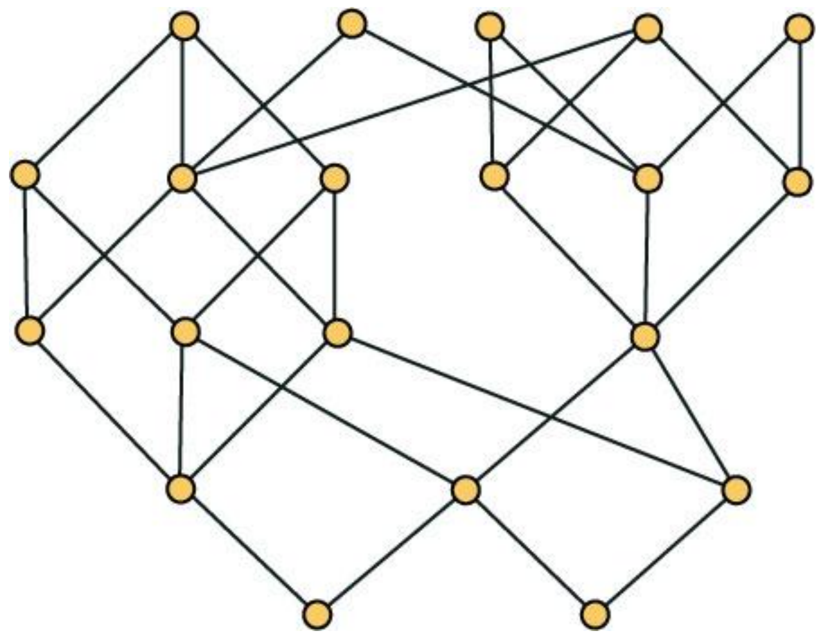
Ranked Posets – Start of Second Proof

Definition A poset is ranked if all maximal chains are maximum. Here is one of height 4.



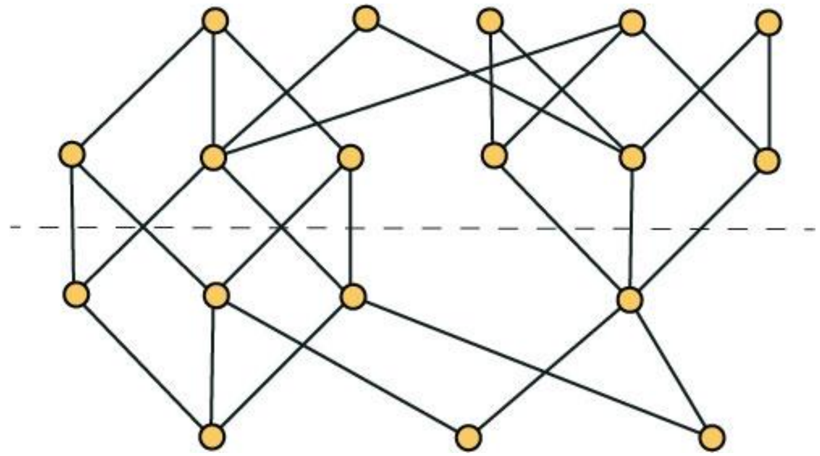
Ranked Posets (2)

Definition A poset is ranked if all maximal chains are maximum. Here is one of height 5.



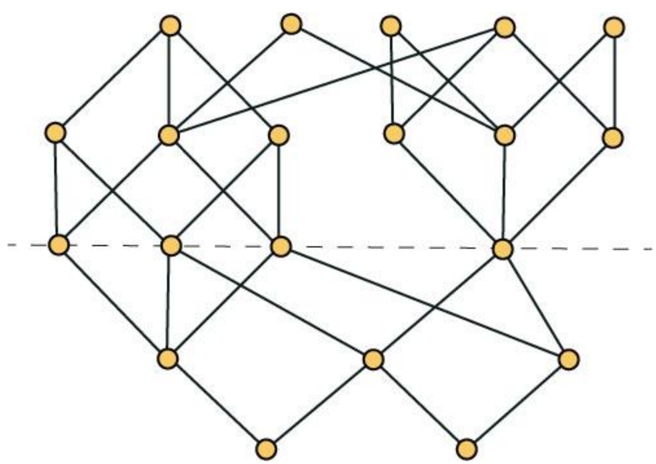
Middle Level of Ranked Poset

Observation Here is the middle level for a ranked poset of height 4.



Middle Level of Ranked Poset (2)

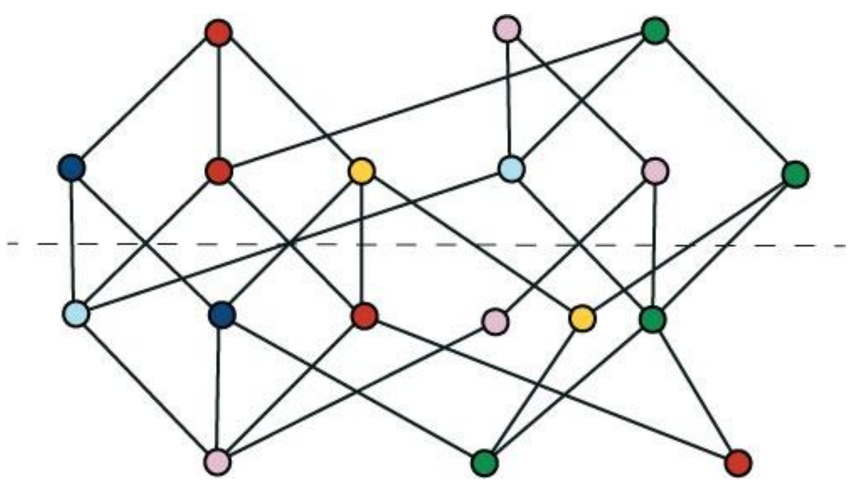
Observation Here is the middle level for a ranked poset of height 5.



Symmetric Chain Partition

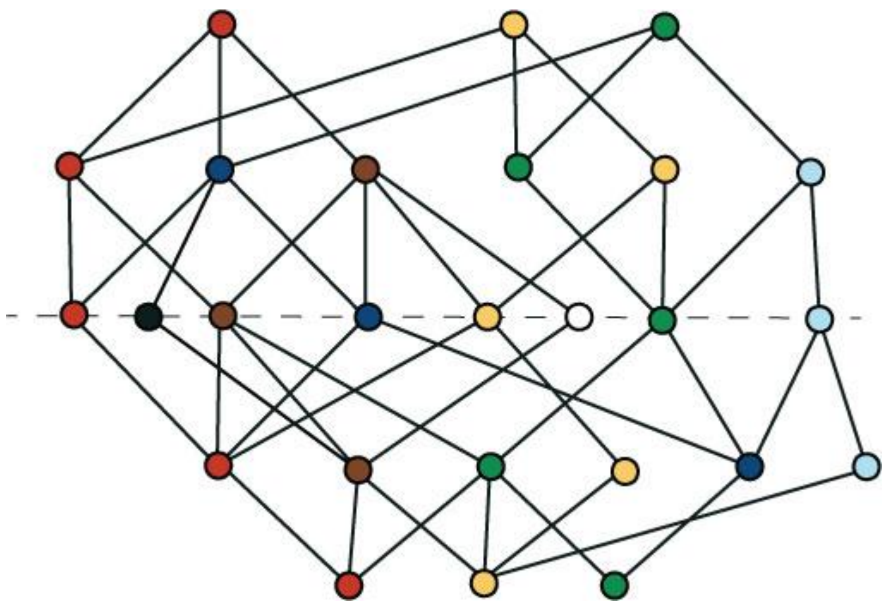
Definition A chain in a ranked poset is symmetric when it (1) goes the same distance above and below the middle levels and (2) doesn't skip levels.

Observation Here is a symmetric chain partition for a ranked poset of height 4.



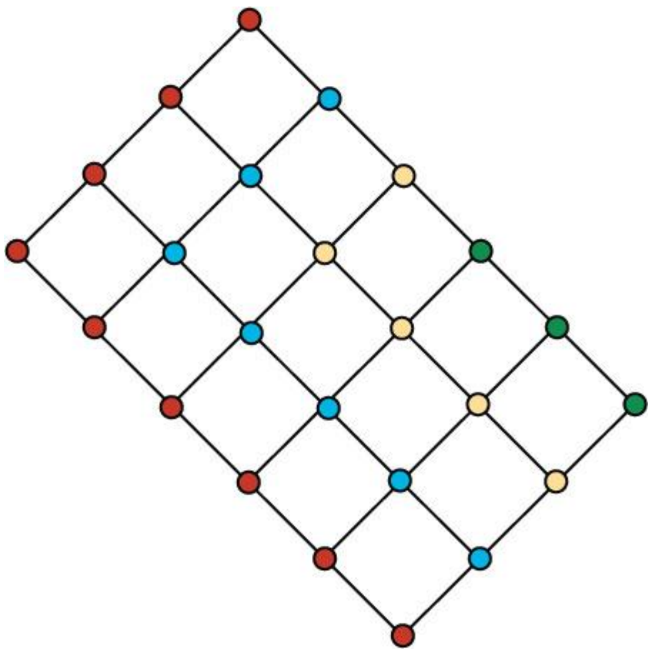
Symmetric Chain Partition (2)

Observation Here is a symmetric chain partition for a ranked poset of height 5.



Symmetric Chain Partition (3)

Lemma The Cartesian product of two chains has a symmetric chain partition.



Symmetric Chain Partition (4)

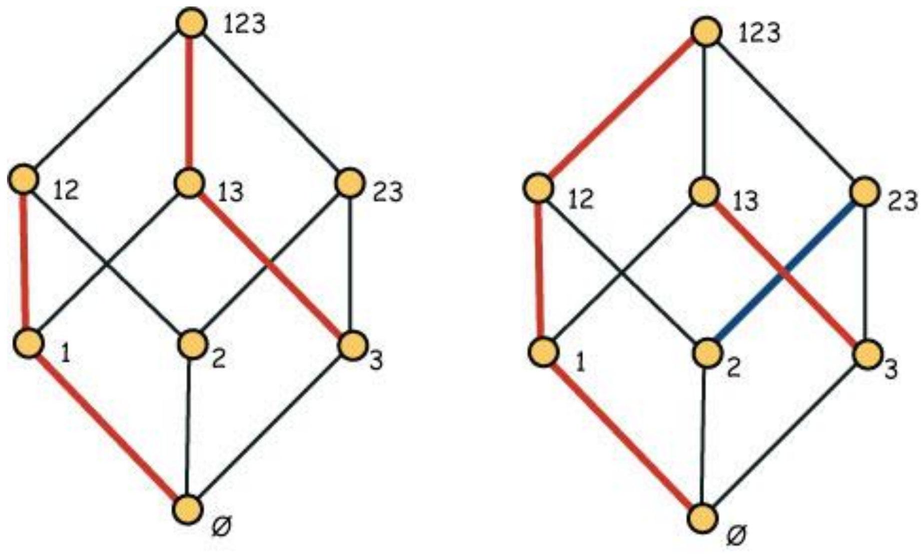
Theorem For every $n \geq 1$, the subset lattice $\mathbf{2}^n$ has a symmetric chain partition.

Proof On the next two slides, we illustrate the inductive construction. In each case, we start with a symmetric chain partition of $\mathbf{2}^n$ and show how to modify two copies to obtain a symmetric chain partition of $\mathbf{2}^{n+1}$.

Note that when n is even, we have some 1-element chains in the partition. Each pair becomes a 2-element chain in the next step. But when n is odd, each pair of chains produces another pair.

Symmetric Chain Partition (5)

Example Using two copies of a symmetric chain partition of $\mathbf{2}^2$ to form a symmetric chain partition of $\mathbf{2}^3$. Note that the 1-element chains 2 and 23 become a 2-element chain.



Symmetric Chain Partition (6)

Example Using two copies of a symmetric chain partition of $\mathbf{2}^3$ to form a symmetric chain partition of $\mathbf{2}^4$.