November 12, 2015

Math 3012 – Applied Combinatorics Lecture 23

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Reminders

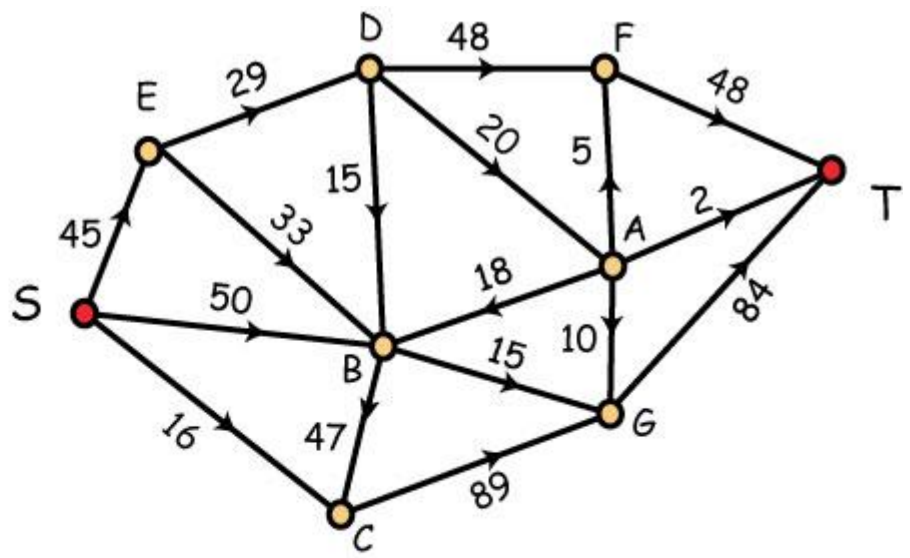
Test 3 Tuesday, November 24, 2015

Final Exam Tuesday, December 8, 2015, 8:05 – 10:55am.

Three-way Option (Full details in email)

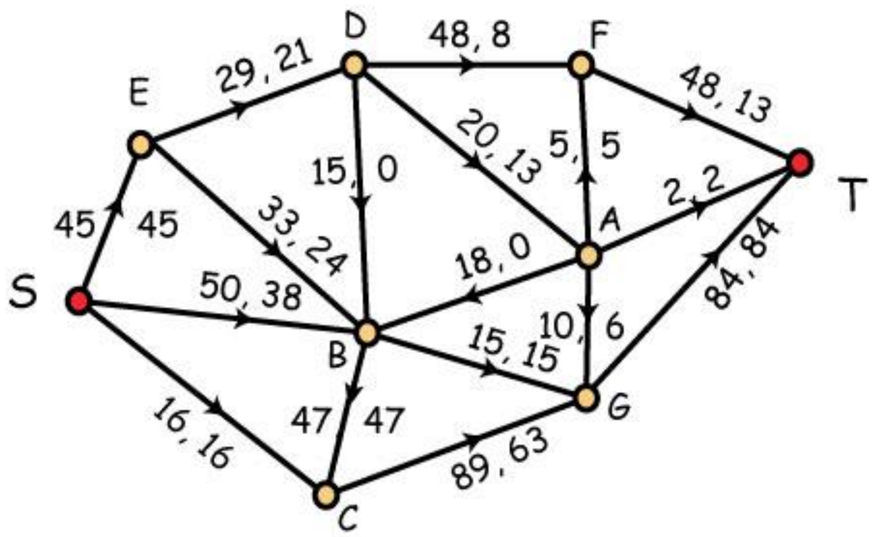
1. Do even numbered problems from assigned set.
2. Obtain/write code for implementing one of the algorithms in our course on my data set.
3. Write 3 – 4 page (typewritten) report on one of the selected math papers, all of which are accessible to undergraduates.

A Network Flow Problem



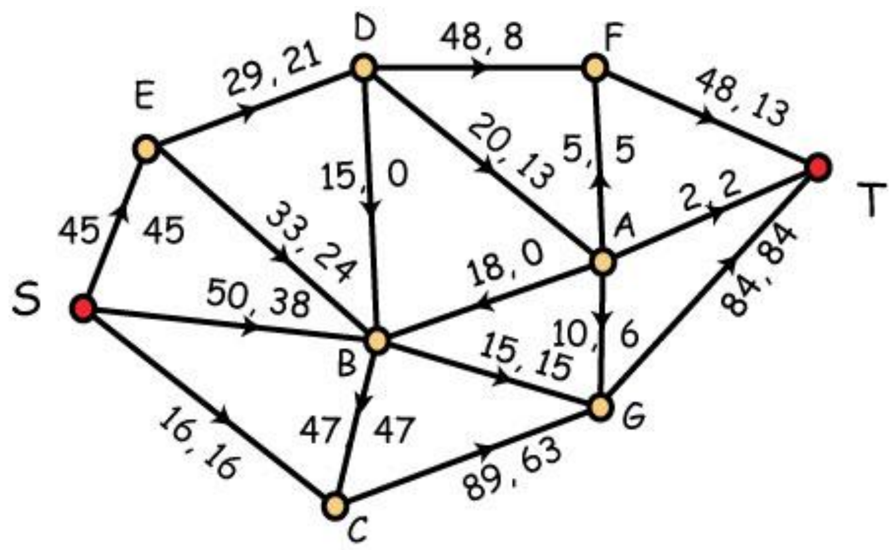
Setup We are given a digraph with positive weights on edges. There are two distinguished vertices, a source S and a sink T. All edges incident with S are oriented away from S and all edges incident with T are oriented towards T. The weights represent capacities and we will denote the capacity of edge e as c_e.

A Flow in a Network



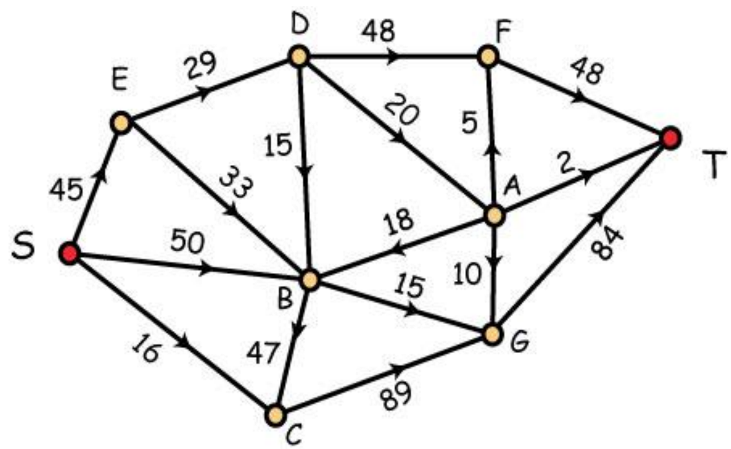
Definition *A* flow is an assignment of a non-negative value x_e to every edge e of the digraph subject to the following conditions: (1) $x_e \le c_e$, i.e., the flow on edge e does not exceed its capacity; (2) the total amount leaving the source = the total amount arriving at the sink; and (3) for all other vertices v, amount into v = amount leaving v.

Challenge: Find a Maximum Flow



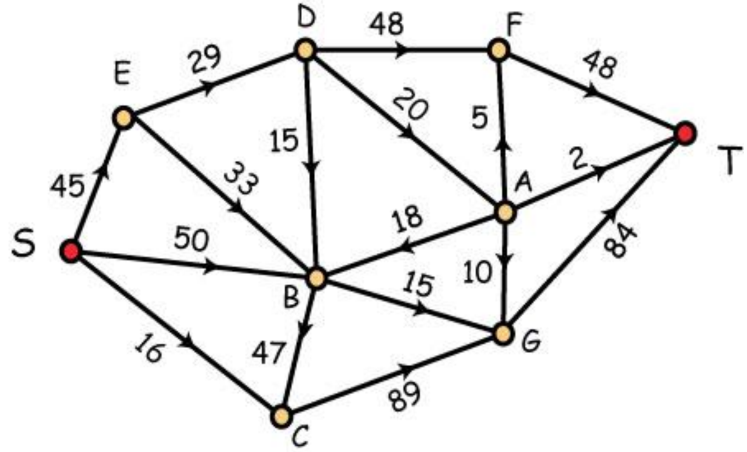
Definition The value of a flow is the amount leaving the source, which is exactly the same as the amount arriving at the sink. This flow has value 45 + 38 + 16 = 99. Our challenge then is to find a maximum flow, i.e., a flow of maximum value. Can you tell by inspection whether this flow is maximum?

Cuts in Network Flow Problems



Definitions A cut in a flow is a partition of the vertices into two subsets L and R with S in **L** and T in **R**. If the network has n vertices, there are 2^{n-2} cuts. The capacity of a cut (**L**, **R**) is the sum of the capacities of all edges from **L** to **R**. Note that we do not include the capacities of edges from **R** to **L** in this sum. Here the capacity of the cut where **L** = (S, E, B, C) and the remaining vertices are in **R** is 29 + 15 + 89 = 123.

The Max Flow/Min Cut Theorem

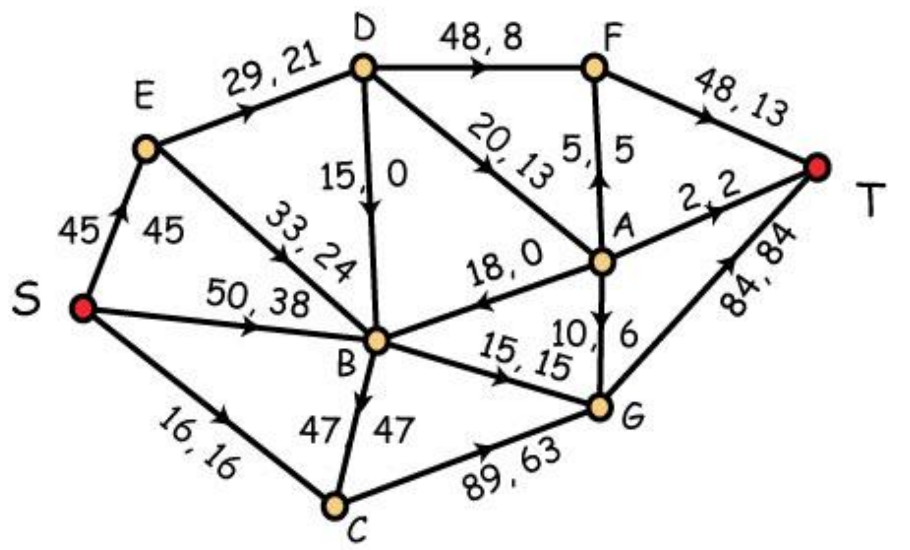


Theorem The maximum value of a flow is equal to the minimum capacity of a cut.

Observation The fact that the value of any flow is at most the capacity of any cut is an immediate (and elementary consequence) of the conservations laws and the definitions of flows and cuts.

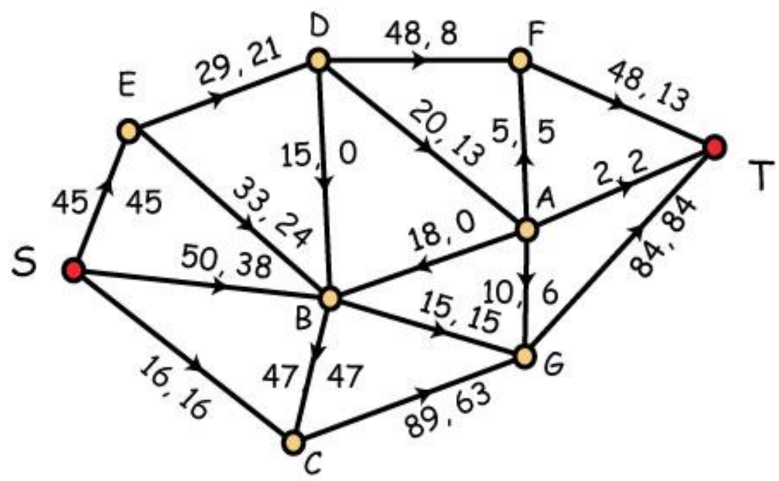
Remark So the challenge in the theorem is finding the maximum flow and the minimum cut, and this is what we will do next!

Full end Empty Edges



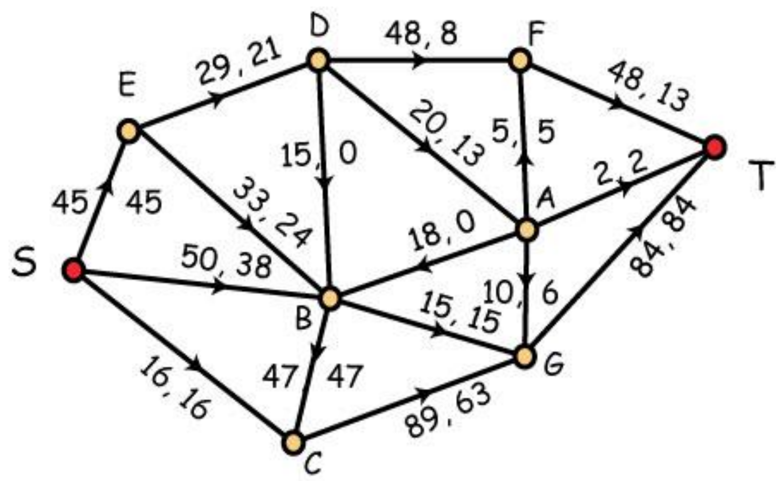
Definitions An edge is full when the flow on the edge is equal to the capacity of the edge. An edge is empty when the flow on the edge is 0. Here edges (S, E), (S, C), (B, G), (A, F), (A, T) and (G, T) are full while (D, B) and (A, B) are empty.

Augmenting Paths



Definitions Allowing the ability to walk on an edge in the network in either direction, an ordinary path from S to T traverses some edges in a forward manner and others in a backwards manner. The first and last are always forward. The path is called an augmenting path when the forward edges are not full and the backwards edges are not empty. Here, the path (S, B, E, D, F, T) is an augmenting path.

Augmenting Paths (2)



Observations A forward edge on an augmenting path has spare capacity and a backwards edge has excess flow. Let v be the minimum value among these quantities. Update the flow by increasing the flow on the forward edges by v and decreasing the flow on the backwards edges by v. For the augmenting path (S, B, E, D, F, T), the quantity v is 8 which is the spare capacity on edge (E, D).

November 17, 2015

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value of the flow is 99 + 8 = 107. In this new flow, do you see any augmenting paths?