KEY

Student Name and ID Number

MATH 3012 Final Exam, Section F, April 27, 2009, WTT

1. Consider the 36-element set consisting of the ten digits $\{0, 1, 2, \ldots, 9\}$ and the twenty-six lower case letters $\{a, b, \ldots, z\}$.

8 $= 4 \times 2$

a. How many strings of length 14 can be formed if repetition of symbols is permitted?

36^{14}

b. How many strings of length 14 can be formed if repetition of symbols is *not* permitted?

$P(36, 14) = \underbrace{36 \cdot 35 \cdot 34 \cdots 23}_{14 \text{ terms}}$

c. How many strings of length 14 can be formed using exactly three 5's, four a's and seven d's?

$\binom{14}{3,4,7} = \frac{14!}{3!\,4!\,7!}$

d. How many strings of length 14 can be formed if exactly three characters are digits and exactly four of the remaining characters are b's?

$\binom{14}{3} 10^3 \binom{11}{4} 25^7$

2. How many integer valued solutions to the following inequalities:

6 $= 3 \times 2$

a. $x_1 + x_2 + x_3 + x_4 < 54$, all $x_i > 0$.   same as $x_1 + x_2 + x_3 + x_4 + x_5 = 54$ all $x_i > 0$

$\binom{53}{4}$

b. $x_1 + x_2 + x_3 + x_4 < 54$, all $x_i \geq 0$.   Add one for x_1, x_2, x_3, x_4. Not x_5.

$\binom{57}{4}$

c. $x_1 + x_2 + x_3 + x_4 < 54$, all $x_i > 0$, $x_3 < 5$.   Give x_5 4 in advance $\binom{49}{4}$ now $x_5 > 0$

$\binom{53}{4} - \binom{49}{4}$

3. Use the Euclidean algorithm to find $d = \gcd(1155, 252)$.

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$252 \overline{)1155}$ quotient 4; 1008; 147

$147 \overline{)252}$ quotient 1; 147; 105

$105 \overline{)147}$ quotient 1; 105; 42

$42 \overline{)105}$ quotient 2; 84; 21

$21 \overline{)42}$ quotient 2; 42; 0

$\boxed{\text{g.c.d}(1155, 252) = 21}$

$21 = 105 - 2 \cdot 42$
$= 1 \cdot 105 - 2(147 - 1 \cdot 105)$
$= 1 \cdot 105 - 2 \cdot 147 + 2 \cdot 105$
$= 3 \cdot 105 - 2 \cdot 147$
$= 3(252 - 1 \cdot 147) - 2 \cdot 147$
$= 3 \cdot 252 - 5 \cdot 147$
$= 3 \cdot 252 - 5(1155 - 4 \cdot 252)$
$= 23 \cdot 252 - 5 \cdot 1155$

4. Use your work in the preceding problem to find integers a and b so that $d = 1155a + 252b$.

5

$\boxed{a = -5 \qquad b = 23}$

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5. For a positive integer n, let t_n count the number of ternary strings of length n that do not contain 112 as a substring (occuring in consecutive positions). Then $t_1 = 3$, $t_2 = 9$ and $t_3 = 26$. Develop a recurrence for t_n and use it to find t_6.

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$\boxed{0}$ t_{n-1}

$\boxed{1}$ t_{n-1}

$\boxed{2}$ t_{n-1} Not all good Those that end 112 are bad

$$t_n = 3t_{n-1} - t_{n-3}$$

$t_4 = 3 \cdot t_3 - t_1 = 3 \cdot 26 - 3 = 78 - 3 = 75$

$t_5 = 3 t_4 - t_2 = 3 \cdot 75 - 9 = 225 - 9 = 216$

$t_6 = 3 \cdot t_5 - t_3 = 3 \cdot 216 - 26 = 648 - 26$

$\boxed{t_6 = 622}$

6. Use the algorithm developed in class to find an Euler ~~circuit~~ path in the graph G shown below:

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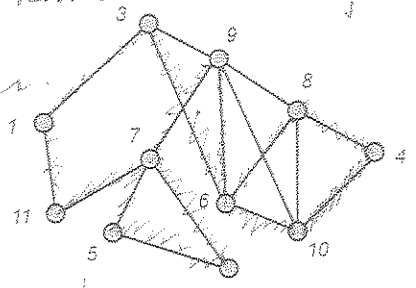
Vertices 3 and 9 have odd degree. All others are even.



~~(1, 3, 6, 8, 4, 10, 6, 9, 7, 2, 5, 7, 11, 1)~~

~~(1, 3, 6, 8, 4, 10, 6, 9, 8, 10, 9, 7, 2, 5, 7, 11, 1)~~

(3, 1, 11, 7, 2, 5, 7, 9, 3, 6, 8, 4, 10, 6, 9, 8, 10, 9)

7. Consider the graph G from the preceding problem. Find the maximum clique size $\omega(G)$ and a set of vertices that form a maximu clique. Then show that $\chi(G) = \omega(G)$. Nevertheless, G is not perfect. Explain why.

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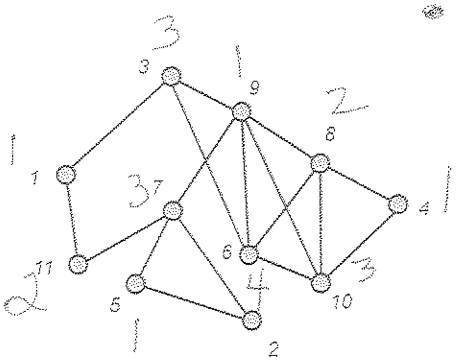
- $\omega(G) = 4$
- $\{6, 8, 9, 10\}$ is a maximum clique



- I have produced a 4-colouring, so $\chi(G) \leq 4$. $\chi(G) \geq 4$ since $\chi(G) \geq \omega(G)$. So $\chi(G) = 4$

- G is not perfect since we would need $\chi(H) = \omega(H)$ for every induced $H \subseteq G$. But vertices 1, 3, 9, 7, 11 form C_5 for which $\chi = 3$ and $\omega = 2$.

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③

8. The graph G in the preceding two problems is not hamiltonian. Identify a non-adjacent pair of vertices xy so that if the edge xy is added to G, the resulting graph is hamiltonian. List the vertices in an order that forms a hamiltonian cycle in this graph.

Add edge $\{2,6\}$ for example

$(1, 3, 9, 8, 4, 10, 6, 2, 5, 7, 11)$

⑤

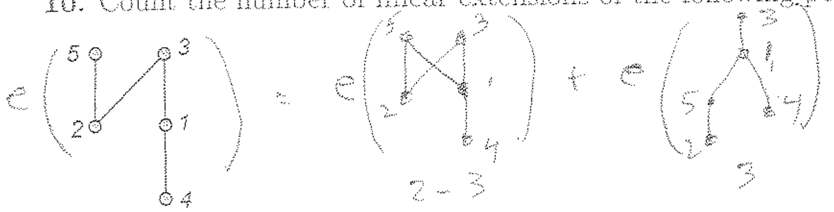
9. How many equivalence relations are there on the set $\{1, 2, \ldots, 59\}$ with class sizes:

$$8, 8, 8, 5, 5, 5, 5, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1$$

$$\frac{59!}{8!\,8!\,8!\,5!\,5!\,5!\,5!\,2!\,2!\,2!\,2!\,2!\,1!\,1!\,1!\,1!\,1!\;\;3!\,4!\,5!\,5!}$$

⑤

10. Count the number of linear extensions of the following poset:



$= 6 + 3 =$ ⑨

④ 4x!

11. For the subset lattice $\mathbf{2}^{10}$,

a. The total number of elements is: 2^{10}

b. The total number of maximal chains is: $10!$

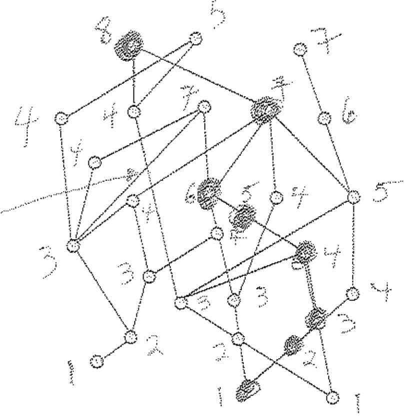
c. The number of maximal chains through $\{1, 4, 5, 8\}$ is: $4!\;6!$

d. The width of $\mathbf{2}^{10}$ is: $\binom{10}{5}$

⑤

12. For the poset P shown below,

take out edge



other possibilities

find the height h and a partition into h minimal elements by recursively stripping off the set of minimal elements. You may display your answer by writing directly on the diagram. Then darken a set of points that form a maximum chain.

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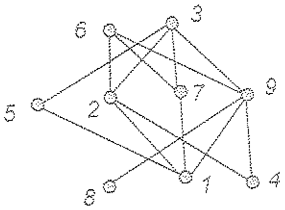
13. The poset P shown below is an interval order:

⑤

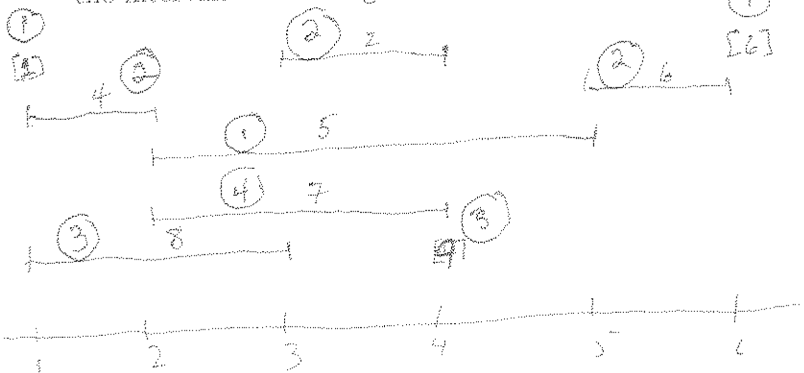


Find the down sets and the up sets. Then use these answers to find an interval representation of P that uses the least number of end points.

①	$D(1) =$ ∅	$U(1) =$ 2, 3, 5, 6, 7, 9	①	$I(1) =$ [1, 1]
③	$D(2) =$ 1, 4	$U(2) =$ 3, 6	④	$I(2) =$ [3, 4]
⑥	$D(3) =$ 1, 2, 4, 5, 7, 8, 9	$U(3) =$ ∅	⑥	$I(3) =$ [6, 6]
①	$D(4) =$ ∅	$U(4) =$ 2, 3, 6, 9	②	$I(4) =$ [1, 2]
②	$D(5) =$ 1	$U(5) =$ 3	⑤	$I(5) =$ [2, 5]
⑤	$D(6) =$ 1, 2, 4, 7, 8, 9	$U(6) =$ ∅	⑥	$I(6) =$ [5, 6]
②	$D(7) =$ 1	$U(7) =$ 3, 6	④	$I(7) =$ [2, 4]
①	$D(8) =$ ∅	$U(8) =$ 3, 6, 9	③	$I(8) =$ [1, 3]
④	$D(9) =$ 1, 4, 8	$U(9) =$ 3, 6	④	$I(9) =$ [4, 4]

In the space below, draw the representation you have found. Then use the First Fit Coloring Algorithm for interval graphs to solve the Dilworth Problem for this poset, i.e., find the width w and a partition of P into w chains. You may display your answers by writing the colors directly on the intervals in the diagram.

⑤



Find a maximum antichain in P:

{4, 5, 7, 8}

14. [a.] Write all the partitions of the integer 10 into odd parts:

10 = 9 + 1
= 7 + 3
= 5 + 5
= ~~7~~ 7 + 1 + 1 + 1
= 5 + 3 + 1 + 1
= 5 + 1 + 1 + 1 + 1 + 1
= 3 + 3 + 3 + 1
= 3 + 3 + 1 + 1 + 1 + 1
= 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1
= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1

10 total

⑥

[b.] Write all the partitions of the integer 10 into distinct parts:

3+2

10 = 10
= 9 + 1
= 8 + 2
= 7 + 3
= 6 + 4
= 7 + 2 + 1
= 6 + 3 + 1
= 5 + 4 + 1
= 5 + 3 + 1
= 4 + 3 + 2 + 1

10 total

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③ [c.] Use generating functions to prove that the number of partitions of an integer into odd parts equals the number of partitions into distinct parts.

G.F. for partitions into odd parts $= \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x^7}\cdots$

G.F. for partitions into distinct parts $= (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\cdots$

~~$\frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot\frac{1}{\cdots} = \frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^2}\cdot\frac{1-x^{6}}{1-x^{3}}\cdot\frac{1-x^8}{1-x^4}$~~

$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\cdots$

$= \frac{\cancel{1-x^2}}{1-x}\cdot\frac{\cancel{1-x^4}}{\cancel{1-x^2}}\cdot\frac{\cancel{1-x^6}}{1-x^3}\cdot\frac{\cancel{1-x^8}}{\cancel{1-x^4}}\left(\frac{1-x^{10}}{1-x^5}\right)\cdots$

$= \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x^7}\cdots$

15. Find the general solution to the advancement operator equation:
$(A+2)^3(A-5)^2(A+4)f(n) = 0$

⑤ $f(n) = c_1(-2)^n + c_2 n(-2)^n + c_3 n^2(-2)^n + c_4 5^n + c_5 n 5^n + c_6(-4)^n$

16. Find the solution to the advancement operator equation:
$(A^2 + 2A - 15)f(n) = 0, \quad f(0) = 11$ and $f(1) = -23$.

$A^2+2A-15 = (A+5)(A-3)$

$f_n = c_1(-5)^n + c_2 3^n$

⑤ $c_1 + c_2 = 11$

$-5c_1 + 3c_2 = -23$

$3c_1 + 3c_2 = 33$

$8c_1 = 56$

$c_1 = 7 \qquad c_2 = 4$

$$\boxed{f(n) = 7(-5)^n + 4\cdot 3^n}$$

17. Write the inclusion/exclusion formula for the number of onto functions from $\{1,2,\ldots,m\}$ to $\{1,2,\ldots,n\}$.

⑤ $$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(\underset{\uparrow\atop n}{\cancel{m}}-i)^m$$

18. Write the inclusion/exclusion formula for the number of derangements on $\{1,2,\ldots,n\}$.

⑤ $$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)!$$

19. Write the inclusion/exclusion formula for the Euler function $\phi(n)$. Then use this formula to evaluate $\phi(180)$.

⑤ $\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_n}\right)$ where $n = p_1^{m_1}p_2^{m_2}\cdots p_n^{m_n}$ is the prime factorization

$180 = 9\cdot 20 = 3^2\cdot 2^2\cdot 5$

$\phi(180) = 180\left(1-\frac{1}{3}\right)\left(1-\frac{1}{2}\right)\left(1-\frac{1}{5}\right) =$ ~~$180\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{4}{5}$~~ 72

$= \cancel{180}^{24}\cdot\frac{2}{\cancel{3}}\cdot\frac{1}{\cancel{2}}\cdot\frac{1}{\cancel{5}} = 8$

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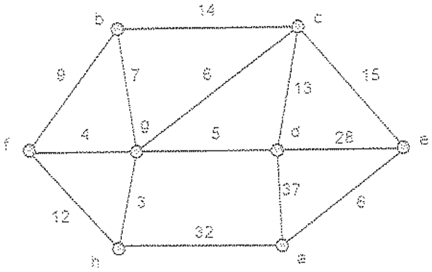
20.

4



In the space below, list *in order* the edges which make up a minimum weight spanning tree using, respectively Kruskal's Algorithm (avoid cycles) and Prim's Algorithm (build tree). For Prim, use vertex a as the root.

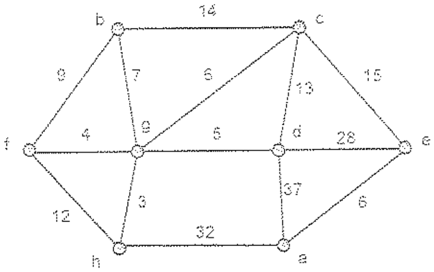
Kruskal's Algorithm

g	h	3
f	g	4
g	d	5
c	g	6
a	e	6
b	g	7
c	e	15

Prim's Algorithm

a	e	6
c	e	15
g	c	6
g	h	3
g	f	4
g	d	5
g	b	7

21.



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For the graph shown above, traffic is allowed to flow in either direction on the edges. Carry out Dijkstra's algorithm to find shortest paths from node a to all other nodes.

	b	c	d	e	f	g	h
	ab ∞	ac ∞	ad 37	ae 6	af ∞	ag ∞	ah 32
scan e	ab ∞	aec 21	aed 34		af ∞	ag ∞	ah 32
scan c	aecb 35		aed 34		af ∞	aecg 37	ah 32
scan h	aecb 35		aed 34		ahf 44	ahg 35	
scan d	aecb 35				ahf 44	ahg 35	
scan b					ahf 44	ahg 35	
scan g					ahgf 39		

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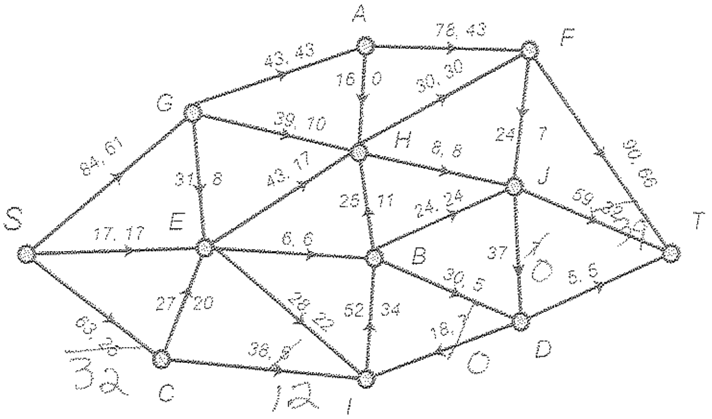
22.



(3) a. What is the current value of the flow? 61 + 17 + 25 = $\boxed{103}$

(3) b. What is the capacity of the cut $V = \{S, G, E, C, I\} \cup \{A, H, B, D, F, J, T\}$.

~~43 + 10 + 17 + 6 + 34~~ ~~110~~

43 + 39 + 43 + 6 + 52

c. Carry out the labeling algorithm, using the pseudo-alphabetic order on the vertices and list below the labels which will be given to the vertices.

(8)

S (*, +, ∞)
C (S, +, 38)
G (S, +, 23)
E (C, +, 7)
I (C, +, 33
H (G, +, 23)

B (I, +, 18)
D (I, −, 7)
J (D, −, 7)
T (J, +, 7)

Augmenting path

f f b b f
S, C, I, D, J, T

(3) d. Use your work in part c to find an augmenting path and make the appropriate changes directly on the diagram.

(6) e. Carry out the labeling algorithm a second time on the updated flow. It should halt without the sink being labeled. Find a cut whose capacity is equal to the value of the flow.

S(*, +, ∞)
C(S, +, 31)
G(S, +, 23)
E(C, +, 7)
I(C, +, 26

H(G, +, 23)
B(I, +, 18)
D(B, +, 18

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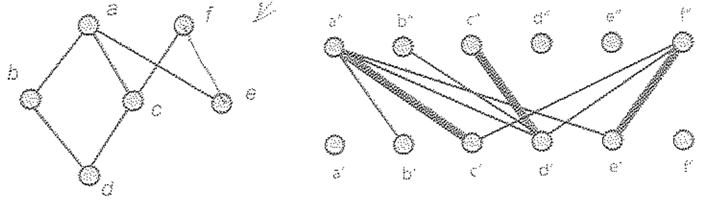
$\mathcal{L} = \{S, C, G, E, I, H, B, D\}$ $\mathcal{U} = \{T, A, F, J\}$

capacity of cut = 43 + 30 + 8 + 24 + 5
= 110 ✓

23. In the figure below, we show a poset and the bipartite graph associated with it. The darkened edges form a maximum matching in the graph. Find the minimum chain partition determined by this matching.

missing line



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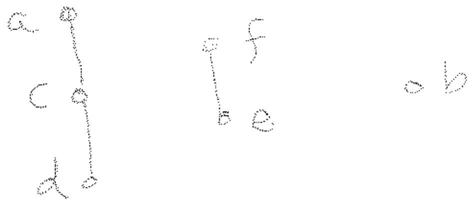

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