Solutions

Student Name and ID Number

MATH 3012, Quiz 3, November 24, 2015, WTT

1. Let n and m be positive integers with $n \geq m$.

a. Write the inclusion/exclusion formula for the number $S(n,m)$ of surjections from $\{1,2,\ldots,n\}$ to $\{1,2,\ldots,m\}$.

8

4+4

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$$S(n,m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$$

b. Evaluate your answer in part a when $n=6$ and $m=4$.

$$S(6,4) = \binom{4}{0}4^6 - \binom{4}{1}3^6 + \binom{4}{2}2^6 - \binom{4}{3}1^6 + \binom{4}{4}0^6$$

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$$= 4096 - 4\cdot 729 + 6\cdot 64 - 4\cdot 1 + 0$$

$$= 4096 - 2916 + 384 - 4$$

$$= 1560$$

2. Recall that $\phi(n)$ is the number of integers in $\{1,2,\ldots,n\}$ which are relatively prime to n. Use inclusion/exclusion to evaluate $\phi(n)$ when $n = 2^3 \cdot 5^2 \cdot 11$.

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$$\phi(n) = 2^3\cdot 5^2\cdot 11\,(1-\tfrac{1}{2})(1-\tfrac{1}{5})(1-\tfrac{1}{11})$$

$$= 2^3\cdot 5^2\cdot 11\ \tfrac{1}{2}\cdot\tfrac{4}{5}\cdot\tfrac{10}{11}$$

$$= 2^2\cdot 5\cdot 4\cdot 10$$

$$= 800$$

3. Recall that $1/(1-x)$ is the closed form for the generating function $1+x+x^2+x^3+x^4+\ldots$. Find the closed form for the generating function: $1 - x/2 + x^2/3 - x^3/4 + x^4/5 - x^5/6 + - \ldots$

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$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 \cdots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$

$$\frac{\ln(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \frac{x^5}{6} + \cdots$$

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4. Write an expression (an infinite product) for the generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ where $a_0 = 1$ and for $n \geq 1$, a_n is the number of partitions of the integer n into distinct parts, all of which are odd. For example $17 = 9 + 5 + 3$ and $44 = 23 + 11 + 9 + 1$.

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$f(x) = (1+x)(1+x^3)(1+x^5)(1+x^7)(1+x^9)(1+x^{11})\cdots$

5. a. Find the general solution to the advancement operator equation: $(A - 2 + i)^3 (A - 5)^2 f(n) = 0$

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$f(n) = c_1 (2-i)^n + c_2 n (2-i)^n + c_3 n^2 (2-i)^n$
$+ c_4 5^n + c_5 n 5^n$

b. Find the general solution to the equation: $(A^2 - 5A + 6) f(n) = 0$.

$A^2 - 5A + 6 = (A-2)(A-3)$

4

$f(n) = c_1 2^n + c_2 3^n$

c. Find a particular solution to the equation: $(A^2 - 5A + 6) f(n) = 14$.

$f(n) = c$

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$(A^2 - 5A + 6) c = c - 5c + 6c = 2c = 14 \Rightarrow c = 7$

$f(n) = 7$ is a particular solution.

d. Find the solution to the equation: $(A^2 - 5A + 6) f(n) = 14$ subject to $f(0) = 6$ and $f(1) = 9$.

$f(n) = c_1 2^n + c_2 3^n + 7$

$c_1 2^0 + c_2 3^0 + 7 = 6 \Rightarrow c_1 + c_2 + 7 = 6$

6

$c_1 2^1 + c_2 3^1 + 7 = 9 \Rightarrow 2c_1 + 3c_2 + 7 = 9$

$c_1 + c_2 = -1$

$2c_1 + 3c_2 = +2$

$2c_1 + 2c_2 = -2$

$c_2 = 4$

$c_1 = -5$

Answer $f(n) = -5 \cdot 2^n + 4 \cdot 3^n + 7$

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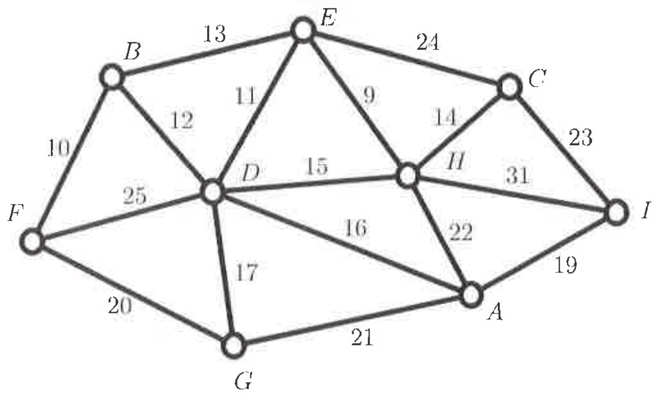
5+5

6. A graph with weights on edges is shown below. In the space to the right of the figure, list *in order* the edges which make up a minimum weight spanning tree using, respectively, Kruskal's Algorithm (avoid cycles) and Prim's Algorithm (build tree). For Prim, use vertex A as the root.



5 **Kruskal**	5 **Prim**
E H	A D
B F	D E
E D	E H
B D	B D
H C	B F
D A	H C
D G	D G
A I	A I

18
4+4+6+6

7. Dijkstra's algorithm is being run on a weighted digraph with vertex set $\{1, 2, \ldots, 8\}$ to find shortest paths from vertex 1 to all other vertices. After 5 iterations, the vertices marked *permanent* are $\{1, 3, 4, 7, 8\}$ and scans have been completed from each of these five vertices. Here are the shortest paths the algorithm has found thus far:

$P(1) = (1)$ total length 0.
$P(8) = (1, 8)$ total length 9.
$P(4) = (1, 4)$ total length 23.
$P(3) = (1, 8, 3)$ total length 24.
$P(6) = (1, 6)$ total length 28.

The candidate paths for the remaining three vertices are:

$P(2) = (1, 4, 2)$ total length 50.
$P(5) = (1, 8, 3, 5)$ total length 44.
$P(7) = (1, 6, 7)$ total length 82.

a. 4 The weight $w(8, 3)$ of the edge $(8, 3)$ is 15. (15 = 24 − 9)

b. 4 The weight $w(6, 7)$ of the edge $(6, 7)$ is 54. = 82 − 28

c. 4 The temporary vertex which is now marked permanent is 5.

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d. Which shortest paths $P(2)$, $P(5)$ and $P(7)$ will Dijkstra find if $w(2, 5) = w(2, 7) = 2$, $w(5, 2) = 4$, $w(5, 7) = 38$ and $w(7, 2) = 1$.

P(5) = (1, 8, 3, 5) total length 44 permanent
Scan from 5
update P(2) = (1, 8, 3, 5, 2) total length 48 = 44+4
candidate P(7) = (1, 8, 3, 5, 7) = 44+38 = 82 reject
P(2) = (1, 8, 3, 5, 2) total length 48 Permanent
update P(7) = (1, 8, 3, 5, 2, 7) total length 50 = 48+2 permanent

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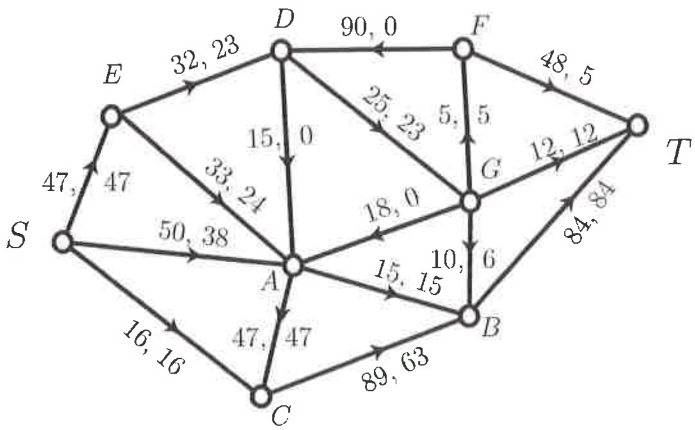
4+4+6+4

8. Consider the following network flow:



a. What is the current value of the flow?

4 $47 + 38 + 16 = 101$

b. What is the capacity of the cut $V = \{S, A, E, C\} \cup \{T, B, D, F, G\}$.

4 $\cancel{32} + 15 + 89 = 136$

c. Carry out the labeling algorithm, using the pseudo-alphabetic order on the vertices and list below the labels which will be given to the vertices. Caution. The labelling should halt without the sink receiving a label.

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S $(*, +, \infty)$
A $(S, +, 12)$
E $(A, -, 12)$
D $(E, +, 9)$
G $(D, +, 2)$
B $(G, +, 2)$
C $(B, -, 2)$

d. Find a cut whose capacity is equal to the value of the current flow.

$\mathcal{L} = \{S, A, B, C, D, E, G\}$ $\mathcal{U} = \{F, T\}$

Note: This cut has capacity $5 + 12 + 84 = 101$

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10x1

9. True–False. Mark in the left margin. Note: The first five of these questions are asked for the application of network flows (and bipartite matchings in particular) to solve the Dilworth problem for a poset P. In these five questions, the symbol G is used to represent the balanced bipartite graph associated with P.

F 1. When x and y are incomparable in P, the edge $x'y''$ is in G.

F 2. For every $x \in P$, the edge $x'x''$ is in G.

F 3. When the labelling algorithm halts and we obtain a maximum matching of size m in G, then we know that the width of P is m.

F 4. When $x'y''$ is an edge in the maximum matching, then x and y belong to distinct chains in the associated chain partition.

T 5. A maximum antichain in P can be obtained by selecting a point x from each chain in the chain partition associated with the maximum matching so that x' is labelled and x'' is unlabelled—when the labelling algorithm halts.

T 6. Let H be a bipartite graph with 250 vertices on one side and 400 on the other side. If the defect of H is 75, there is a matching of size 175 in H.

F 7. To implement Kruskal's algorithm, it is not necessary to sort the edges by weight. One can simply take the edges in any order and take the first one avoiding a cycle when added to those edges already chosen.

F 8. Dijkstra's algorithm finds shortest paths having the maximum number of edges.

F 9. The key idea behind the Ford-Fulkerson algorithm for network flows is to find at each step an augmenting path which maximizes the increase in the amount of the flow.

F 10. All linear programming problems posed with integral constraints have integral solutions.

FUN 11. Weakly convergent generating functions spanning Dilworth partitions admit Kruskal flows with irrational coefficients having distinct odd arrays.

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